CANONICAL FORMS FOR MAPPINGS OF VECTOR SPACES

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In a recent paper [this MONTHLY, vol. 59, 1952, pp. 219–222]* some results were presented illustrating the way in which certain parts of the theory of matrices may be simplified by regarding them as mappings of vector spaces. In the present paper these methods are to be applied to the construction of certain canonical forms to which matrices may be reduced by transformations of type $T^{-1}AT$ (*similarity* transformations). The only added information needed here is some knowledge of the properties of the set $F[x]$ of all polynomials with coefficients in a field F; namely, (1) every two polynomials $\alpha, \beta \in F[x]$ have a greatest common divisor (α, β), (2) there exist polynomials γ, $\delta \in F[x]$ such that $\gamma\alpha + \delta\beta = (\alpha, \beta)$, and (3) every polynomial is uniquely factorable into a finite set of prime (irreducible in $F[x]$) polynomials.

Let A be a linear mapping of an n-dimensional vector space S over a field F. Define $A^0 = I$ (the identity mapping), and consider the successive powers $A^i (i = 0, 1, \cdots)$. If $V \in S$ for which $VA^{i-1} = 0$, then $VA^i = 0$, so that the null space $N(A^{i-1}) \subseteq N(A^i)$. If $\nu(A)$ designates the dimension $d[N(A)]$, this implies that $\nu(A^{i-1}) \leqq \nu(A^i)$. Define $w_i = \nu(A^i) - \nu(A^{i-1})$.

LEMMA 1. *$w_i \leqq w_{i-1}$ for all $i > 1$.*

Proof. The image of S under the mapping A^{i-1} is SA^{i-1}. Thus $SA^i = SA(A^{i-1}) \subseteq SA^{i-1}$. The null space $N(A^i)$ maps under A^{i-1} into a subspace of SA^{i-1}; in fact, into that part of SA^{i-1} which goes into zero under an additional mapping by A. The image of $N(A^i)$ is thus $N(A) \cap SA^{i-1}$. It follows [Lemma 1, p. 220] that $\nu(A^i) = \nu(A^{i-1}) + d[N(A) \cap SA^{i-1}]$. But $N(A) \cap SA^{i-1} \subseteq N(A) \cap SA^{i-2}$, and hence $\nu(A^i) - \nu(A^{i-1}) \leqq \nu(A^{i-1}) - \nu(A^{i-2})$.

A set of vectors $\{V_i\} \in N(A^i)$ is said to be *j-independent* if it forms the basis of a subspace S' of $N(A^i)$ which has zero intersection with $N(A^{i-1})$. (That is, S' is a subspace of the complement of $N(A^{i-1})$ in $N(A^i)$.) Alternatively, j-independence might be defined by requiring that the image of the set $\{V_i\}$ under the mapping A^{i-1} shall be independent. (In the language of cosets, this is the same as saying that the cosets of the $\{V_i\}$ modulo $N(A^{i-1})$ are independent.)

From either form of the definition, a j-independent set must also be independent in the usual sense. In fact, it is clear that if $\{V_i\}$ and $\{W_i\}$ are respectively j-independent and k-independent $(j \neq k)$, then the whole set $\{V_i, W_i\}$ is independent.

* References to this paper will be enclosed in brackets.

LEMMA 2. *For each $j>0$ there exists a set containing w_j j-independent vectors, and no such set contains more than w_j vectors.*

Proof. The maximum number of j-independent vectors is clearly the dimension of the image of $N(A^j)$ under the mapping A^{j-1}. But this is precisely w_j [Lemma 1, p. 220].

Note that an easy way to obtain a j-independent set is first to start with a basis for $N(A^{j-1})$. Then the extra vectors needed to form a basis for $N(A^j)$ are automatically j-independent.

The sum of two mappings A_1 and A_2 is defined in a natural way: for any $V\in S$, $V(A_1+A_2)=VA_1+VA_2$. For a mapping A, let $F[A]$ be the set of all polynomials in A with coefficients in F. It is easily seen that $F[A]$ is isomorphic with the set $F[x]$ of all polynomials in an indeterminate x, under the correspondence $A\rightarrow x$. Thus in the following, we may adopt the nomenclature of $F[x]$, saying, for example, that a polynomial $\alpha=p(A)$ is prime if the corresponding polynomial $p(x)$ is prime (irreducible in $F[x]$), or that one member of $F[A]$ divides another if the corresponding polynomials have that relation in $F[x]$, and so on. Now let $\alpha\in F[A]$ be a prime polynomial of degree h which is also a singular mapping (it will be shown later that such a singular polynomial always exists). Since $N(\alpha)\neq 0$, the w_i relative to α are not all zero. Also, since $\nu(\alpha^j)=\sum_1^j w_i\leqq n$, the set of non-zero w_i is finite. There must thus exist some $w_j\neq 0$, with $w_{j+1}=0$. Then, by Lemma 1, $w_i=0$ for all $i>j$.

To illustrate the process to be used in constructing a canonical form for A, and to avoid complications, we will consider a special case: assuming that, relative to α, $w_3\neq 0$, $w_i=0 (i>3)$. The extension of the method to the general case will be obvious.* Without loss of generality, it may be assumed that α is monic (leading coefficient 1), so that

$$\alpha^3 = A^{3h} + a_{3h}A^{3h-1} + \cdots + a_1 I. \tag{1}$$

Since $w_3\neq 0$, there exists a vector $V\in N(\alpha^3)$, $V\notin N(\alpha^2)$. Suppose $\beta\in F[A]$ such that $V\beta=0$. Let α^t be the largest power of α contained in β, then $\beta=\alpha^t\beta'$. But since α is prime, this implies that $(\beta',\alpha)=1$, and so β' is relatively prime to any power of α. There thus exist $\gamma,\delta\in F[A]$ for which $\beta'\gamma+\alpha^3\delta=1$. (In the language of congruences, $\beta'\gamma\equiv 1 \pmod{\alpha^3}$.) The mapping γ applied to $V\beta$, using the fact that any two members of $F[A]$ are commutative, thus yields $V\alpha^t=0$. Since $V\notin N(\alpha^2)$, it follows that $t\geqq 3$, and so β is of degree $\geqq 3h$. It follows that the set $V_i=VA^{i-1}$ $(i=1,\cdots,3h)$ is independent, and is therefore the basis of a $3h$-dimensional subspace S_1. Now $V_iA=V_{i+1} (i=1,\cdots,3h-1)$, and using (1), $V_{3h}A=-\sum_1^{3h}a_iV_i$. Accordingly S_1 is *invariant* under A (that is, A maps S_1 into itself), the matrix of the mapping, relative to the basis $\{V_i\}$, being

* Some of the ideas used here may also be found in B. L. van der Waerden, *Moderne Algebra*, vol. 2, Springer Berlin, 1931, pp. 135–139. See also C. C. MacDuffee, *Vectors and Matrices*, Carus Mathematical Monographs, No. 7, 1943, Chapter VI; and Paul R. Halmos, *Finite Dimensional Vector Spaces*, Princeton University Press 1942, Appendix I, pp. 159–169.

$$(2) \qquad \begin{bmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 1 \\ -a_1 & \cdot & \cdot & \cdot & -a_{3h} \end{bmatrix}.$$

This is the so-called *companion matrix* of the polynomial α^3. A space for which such a basis exists will be said to be *associated* with α^3.

If for some $\beta \in F[A]$, $V\beta$ (with $V \in S_1$) maps into zero under α^2, then clearly α divides β, and so the set $\{V_i\}$ $(i=1, \cdots, h)$ is 3-independent. By Lemma 2, this means that $w_3 \geqq h$. Similarly $\{V_i\alpha\}$ and $\{V_i\alpha^2\}$ $(i=1, \cdots, h)$ are respectively 2-independent and 1-independent. Accordingly, the whole set $\{V_i, V_i\alpha, V_i\alpha^2\}$ is independent and is therefore a basis for S_1. Further, the space $S_1\alpha^2$ has dimension h, and thus at most h members of S_1 are 3-independent. It may be of interest to note that relative to the basis $\{V_i, V_i\alpha, V_i\alpha^2\}$, the matrix of the mapping of S_1 assumes another well-known canonical form.*

If $w_3 > h$, then by Lemma 2 there exists a vector W such that the set $\{V_i, W\}$ $(i=1, \cdots, h)$ is 3-independent. As in the above discussion, a vector space S_2 may then be set up with basis $W_i = WA^{i-1} (i=1, \cdots, 3h)$, again mapping under A with companion matrix (2). This space also has alternative basis $\{W_i, W_i\alpha, W_i\alpha^2\}$ $(i=1, \cdots, h)$.

Now suppose it were possible to have $V\beta_1 + W\beta_2 \in N(\alpha^2)$ for some $\beta_1, \beta_2 \in F[A]$ each of degree $< h$. Since α is prime there exists $\gamma \in F[A]$ such that $\beta_1\gamma \equiv 1 \pmod{\alpha^3}$. Since $N(\alpha^2)$ is closed under $F[A]$ and $W \in N(\alpha^3)$, mapping by γ yields the relation $V\beta_1\gamma + W \in N(\alpha^2)$. But $V\beta_1\gamma \in S_1$ and so is expressible in terms of $\{V_i\}$ $(i=1, \cdots, h)$ together with a vector in $N(\alpha^2)$. The set $\{V_i, W\}$ would thus not be 3-independent, contrary to the choice of W. Accordingly the whole set $\{V_i, W_i\}$ $(i=1, \cdots, h)$ is 3-independent, and so also $w_3 \geqq 2h$. Similarly the sets $\{V_i\alpha, W_i\alpha\}$ and $\{V_i\alpha^2, W_i\alpha^2\}$ are respectively 2-independent and 1-independent. The whole set made up of these $6h$ vectors is therefore independent, and since it spans $S_1 + S_2$, it is a basis for that space. Further, its dimension $6h = d[S_1] + d[S_2]$, and so [Lemma 2, p. 221] $S_1 \cap S_2 = 0$.

The above process may evidently be continued as long as w_3 holds out, from which it follows that $w_3 = w_3'h$ for some integer w_3'. In the course of the process *disjoint* spaces (that is, spaces each having zero intersection with the sum of all the rest) $S_k (k=1, \cdots, w_3')$ are defined, each associated with α^3. The sum $S' = \sum_1^{w_3'} S_k$ is then a $3w_3$-dimensional space one of whose bases is made up of w_3 3-independent vectors, and a similar number of 2-independent and 1-independent vectors. Further, each S_k is invariant under A (each associated with α^3), and thus the matrix of the mapping of S' consists of blocks of submatrices down the main diagonal (direct sum), each the companion matrix of α^3.

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* Cf. MacDuffee, *loc. cit.*, p. 127.

Now the space S' constructed above contains just w_3 2-independent vectors. Thus if $w_2 > w_3$, there must by Lemma 2 exist at least one vector which forms with them a larger 2-independent set. In a manner similar to the above, this vector then determines a $2h$-dimensional subspace associated with α^2. By a proof parallel to that above, the new space has zero intersection with S', and adds h vectors each to the sets of 2-independent and 1-independent vectors. This process may be continued in turn, showing that $w_2 = w_2' h$ for some integer w_2', and defining in the process $w_2' - w_3'$ disjoint spaces each associated with α^2. Finally, if $w_1 > w_2$, then $w_1 = w_1' h$ and there exist an additional $w_1' - w_2'$ h-dimensional spaces each associated with α. The whole set of spaces defined above is disjoint and so their sum is a space whose basis may be chosen to be the combined set of their individual bases. Its dimension is $\sum_1^3 w_i = \nu(\alpha^3)$, and so it is the null space $N(\alpha^3)$. Each of its constituent subspaces is invariant under A, with matrix the companion matrix of the appropriate power of α, and so, relative to the basis chosen, the matrix of the mapping for $N(\alpha^3)$ is the direct sum of these companion matrices.

A set of disjoint subspaces of a given space, whose sum is equal to the given space is called a *decomposition* of the space. The above construction, as applied to the general case, may be formalized as

THEOREM 1. *Let A be a linear mapping of a finite dimensional vector space, and α a singular prime polynomial in A of degree h. Relative to α, each $w_i = w_i' h$ for some integer w_i'. For some integer j, $w_j \neq 0$ and $w_i = 0$ $(i > j)$; then $N(\alpha^i)$ has a decomposition into invariant subspaces, $w_i' - w_{i+1}'$ of which are associated with α^i. The totality of the bases of its constituent subspaces is a basis for $N(\alpha^i)$, and relative to this basis, $N(\alpha^i)$ maps under A with matrix the direct sum of the matrices of the subspaces.*

Now suppose $\alpha_1, \alpha_2 \in F[A]$ with $(\alpha_1, \alpha_2) = 1$. If $V \in N(\alpha_1) \cap N(\alpha_2)$, then $V\alpha_2 = 0$, while $\alpha_2 \beta \equiv 1 \pmod{\alpha_1}$ for some mapping β. Since also $V\alpha_1 = 0$, it follows that $V = 0$, and so $N(\alpha_1) \cap N(\alpha_2) = 0$. If α_1 and α_2 are singular prime polynomials, the construction of Theorem 1 relative to α_1 and α_2 thus results in disjoint subspaces whose sum maps under A with matrix which is the direct sum of the companion matrices of its constituent subspaces, and whose dimension is the sum of their dimensions. The set of all prime singular polynomials is therefore finite, for each contributes at least one to the dimension of the sum of the associated subspaces, and the whole dimension cannot exceed n.

To show that the set of singular prime polynomials is non-empty, consider first the set of all matrices. This is itself a vector space with basis $\{A_{ij}\}$, where A_{ij} is a matrix with 1 in the i, j position and 0 elsewhere. Since this space has dimension n^2, it follows that no more than n^2 of the matrices $A^i (i = 0, 1, \cdots)$ may be independent. There must thus exist some polynomial (of degree $\leqq n^2$) $\gamma = 0$, so that $\nu(\gamma) = n$. But γ may be factored into a finite set of prime factors at least one of which must have nullity greater than zero, since the nullity of their product is not zero.

Let $\{\alpha_i\}$ be the set of all distinct singular prime polynomials. For each α_i a power exists such that no higher power of α_i has greater nullity; let β_i be the least such power of α_i. Since $N(\beta_i)\cap N(\beta_j)=0 (i\neq j)$, it follows [Lemma 1, p. 220] that $\nu(\prod\beta_i)=\sum\nu(\beta_i)$. The β_i are commutative, so the sum $\sum N(\beta_i)\subseteq N(\prod\beta_i)$, and the equality of their dimensions implies that $\sum N(\beta_i)=N(\prod\beta_i)$. The sum of the subspaces constructed for the various α_i in accordance with Theorem 1 thus form a basis for $N(\prod\beta_i)$. Now let $\{\gamma_i\}$ be the set of all singular prime power factors of γ. But $\{\alpha_i\}$ is the complete set of all distinct singular prime polynomials, and thus each γ_i must be a power of some one of them, say α_i. It follows that $\nu(\gamma_i)\leqq\nu(\beta_i)$, and hence the result $n=\nu(\gamma)=\sum\nu(\gamma_i)\leqq\sum\nu(\beta_i)$. Thus $\nu(\prod\beta_i)=n$. This establishes

THEOREM 2. *The subspaces constructed in accordance with Theorem* 1 *for the various singular prime polynomials of a mapping A define a decomposition of the whole space S into invariant subspaces each associated with the appropriate power of the polynomial. The totality of bases defined in the construction form a basis for S, and relative to this basis the matrix of A is the direct sum of the companion matrices of the constituent subspaces.*

Now if A' is the matrix of the mapping expressed relative to some other basis, it is well known that there is a non-singular matrix T such that $T^{-1}A'T=A$. Theorem 2 may therefore be rephrased to state that every matrix is similar to a direct sum of companion matrices. Further, the set of companion matrices depends only on the various sets of $\{w_j\}$, and these depend only on the nullities of the powers of the corresponding α_i. Since the nullity of a mapping is invariant under change of basis, it follows that this form is unique (except for the order in which the constituent companion matrices are chosen).

It may be remarked that if $\alpha=A-cI$ is singular (in which case c is called a *characteristic value*), then the set $\{w_i\}$ relative to α is called the *Weyr characteristic* of A relative to c. It is easy to show that a slight variation of the above construction then leads to the so-called Jordan matrix. For example, suppose $w_j\neq 0$ while $w_i=0 (i>j)$. The basis of the first subspace for α^i may then be chosen as follows: Let $V_1\in N(\alpha^i)$ such that $V_1\notin N(\alpha^{i-1})$, and then define $V_{i+1}=V_i\alpha$ $(i=1,\cdots,j-1)$. Thus $V_iA=cV_i+V_{i+1} (i=1,\cdots,j-1)$ and $V_jA=cV_j$. Relative to the basis $\{V_i\}$ the mapping thus has the matrix (Jordan Matrix)

$$\begin{bmatrix} c & 1 & 0 & \cdot & \cdot & 0 \\ 0 & c & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & c & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & c \end{bmatrix}.$$

If all the singular prime polynomials are linear, then A is evidently similar to the direct sum of the appropriate Jordan matrices (Jordan normal form).

ONE-SIDED MAXIMA AND MINIMA OF FUNCTIONS OF TWO OR MORE VARIABLES

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1. Introduction. We shall first consider a real single-valued function

$$z = f(x, y)$$

of the two real variables x and y, which is continuous on a closed region R bounded by an ordinary* curve C. It is well known that such a function has an absolute maximum value and an absolute minimum value in the region R. These absolute extremes, as well as any relative extremes, of the function $f(x, y)$ may be at points interior to the region R or at points of its boundary C. In the latter case, the extreme is said to be *one-sided* (unilateral). So far as the author knows there is no reference in the literature to one-sided extremes of functions of two or more variables.

Before treating one-sided extremes, we shall determine necessary conditions and sufficient conditions in order that $f(x, y)$ have an extreme value at a point (x_0, y_0) interior to R in terms of cylindrical coordinates. The methods used are simplifications of those of Van Dantscher.§ Next, we shall determine necessary conditions and sufficient conditions in order that $f(x, y)$ have an extreme value at a point (x_0, y_0) on the boundary C of the region R of admissible points, under the assumption that appropriate continuity and differentiability properties of $f(x, y)$ hold on an extended region R' containing R in its interior.

Finally, we shall show how the results of the general case of one-sided extremes are easily extended to functions of three or more variables. One reason for treating first the case of two variables in detail is that certain special cases can be fully developed for functions of two variables which do not seem to extend, by the methods employed, to functions of three or more variables.

2. Free relative extremes. Suppose that the point to be investigated has the coordinates (x_0, y_0) and lies interior to the region R. Let

$$x - x_0 = r \cos\theta, \qquad y - y_0 = r \sin\theta,$$

where r and θ are restricted to the region

$$0 \leqq r \leqq d, \qquad 0 \leqq \theta \leqq 2\pi,$$

for $d > 0$. Then, we have

$$z = f(x_0 + r\cos\theta, y_0 + r\sin\theta) \equiv z(r, \theta).$$

If $f(x, y)$ has a minimum at (x_0, y_0), then by definition there exists a value of d

* By an "ordinary curve" is meant one which is either of class C' or is made up of a finite number of arcs of class C'.

§ *Cf.* Math. Ann., vol. 42, p. 89. Also, Hancock, Theory of Maxima and Minima, Ginn and Co., 1917, pp. 42–69.

such that

$$z(r, \theta) - z(0, \theta) \geqq 0,$$

on the above region. Therefore, we may state

THEOREM 1. *If $f(x, y)$ is continuous in R and assumes a minimum at (x_0, y_0) interior to R, and if*

$$z_{r+}(0, \theta) = \lim_{r=0+} [z(r, \theta) - z(0, \theta)]/r$$

exists for $0 \leqq \theta \leqq 2\pi$, then

$$z_{r+}(0, \theta) \geqq 0, \qquad 0 \leqq \theta \leqq 2\pi.$$

If $f(x_0, y_0)$ is a maximum then

$$z_{r+}(0, \theta) \leqq 0, \qquad 0 \leqq \theta \leqq 2\pi.$$

THEOREM 2. *A continuous function $f(x, y)$ assumes a minimum at (x_0, y_0) in R if*

$$\lim_{r=0+} [z(r, \theta) - z(0, \theta)]/r = z_{r+}(0, \theta) \geqq \rho > 0, \qquad 0 \leqq \theta \leqq 2\pi,$$

uniformly in θ. If the first two inequalities in the preceding line are reversed, then $f(x_0, y_0)$ is a maximum.

As an example, let us consider the function

$$z = \sqrt{x^2 + 2y^2}$$

and the point (0, 0). In this case we have

$$z(r, \theta) = |r| \sqrt{1 + \sin^2 \theta},$$

and

$$z_{r+}(0, \theta) = \sqrt{1 + \sin^2 \theta} \geqq 1 > 0$$

uniformly for all values of θ. Therefore, $f(0, 0)$ is a minimum which is also an absolute minimum for all values of x and y.

If $f(x, y)$ is of class C' in R, then

$$z_r(r, \theta) = f_x \cos \theta + f_y \sin \theta$$

in the region

$$N: \quad |r| \leqq d, \qquad 0 \leqq \theta \leqq 2\pi.$$

Consequently, $z_r(0, \theta) = 0$, $0 \leqq \theta \leqq 2\pi$ if and only if $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, and we may state

THEOREM 3. *If $f(x, y)$ is of class C' in R and assumes a maximum or a minimum*

at (x_0, y_0) *in* R, *then*

$$z_r(0, \theta) = 0, \qquad 0 \leqq \theta \leqq 2\pi.$$

If $f(x, y)$ is of class C^{n+1} in R, then

$$z_r^{(k)}(r, \theta) = (\cos\theta\,\partial/\partial x + \sin\theta\,\partial/\partial y)^k f(x, y), \qquad k = 1, 2, \cdots, n+1,$$

and $z_r^{(k)}(0, \theta) = 0$, $0 \leqq \theta \leqq 2\pi$, if and only if all derivatives of $f(x, y)$ of order k are zero at (x_0, y_0). If all derivatives of order less than n vanish at (x_0, y_0) while not all of order n vanish there, it follows from Taylor's development that

$$\begin{aligned}(1)\qquad \Delta z = f(x_0 + r\cos\theta, y_0 + r\sin\theta) - f(x_0, y_0) &= r^n[z_r^{(n)}(0, \theta) + rL_n]/n! \\ &= r^n\phi(r, \theta)/n!\end{aligned}$$

where L_n is finite for $|r|$ sufficiently small, since $f(x, y)$ is assumed to be of class C^{n+1}, and where

$$(2)\qquad z_r^{(n)}(0, \theta) = (\cos\theta\,\partial/\partial x + \sin\theta\,\partial/\partial y)^n f(x_0, y_0).$$

We seek conditions which insure the existence of a value of d such that Δz is of one sign in the region N. Obviously the factor $r^n/n!$ may be omitted, since $r = 0$ corresponds to (x_0, y_0). We are interested in the lower limit of the absolute values of the real roots of the equation

$$\phi(r, \theta) = 0, \qquad 0 \leqq \theta \leqq 2\pi.$$

If this lower limit is different from zero, then $f(x_0, y_0)$ is an extreme, but if it is zero then $f(x_0, x_0)$ may or may not be an extreme. There are three cases to be considered.

Case I. If the function (2) is different from zero on $0 \leqq \theta \leqq 2\pi$, which can happen only if n is even, then it follows from (1) that $f(x_0, y_0)$ is an extreme. Consequently, we may state

THEOREM 4. *If* (x, y) *is of class* C^{2n+1} *in* R *and at* (x_0, y_0)

$$z_r^{(k)}(0, \theta) = 0, \quad k = 1, 2, \cdots, 2n-1, \qquad z_r^{(2n)}(0, \theta) \neq 0, \quad 0 \leqq \theta \leqq 2\pi,$$

then $f(x_0, y_0)$ *is a minimum if* $z_r^{(2n)}(0, \theta) > 0$ *and a maximum if* $z_r^{(2n)}(0, \theta) < 0$.

The reader's attention is called to the close analogy between this theorem and the corresponding one for functions of a single variable.

Case II. If the function (2) assumes both positive and negative values on $0 \leqq \theta \leqq 2\pi$, as for example when n is odd, then it follows from (1) that Δz assumes both positive and negative values in every sufficiently small neighborhood of (x_0, y_0). Thus, we have

THEOREM 5. *If* $f(x, y)$ *is of class* C^{n+1} *in* R *and*

$$z_r^{(k)}(0, \theta) = 0, \qquad k = 1, 2, \cdots, n-1, 0 \leqq \theta \leqq 2\pi,$$

while $z_r^{(n)}(0, \theta)$ *assumes both positive and negative values on* $0 \leqq \theta \leqq 2\pi$, *then* $f(x_0, y_0)$ *is neither a maximum nor a minimum.*

Case III (*Ambiguous case*). If the function (2) vanishes for certain values of θ but does not change sign, which can happen only if n is even, then we are not able to decide from (2) whether $f(x_0, y_0)$ is an extreme or not. The ambiguity can usually be removed by the following procedure.

Suppose that the function $z_r^{(n)}(0, \theta)$ in this case vanishes for $\theta = \theta_t$, $t = 1, 2, \cdots, m$, and expand

$$z(r, \theta) = f(x_0 + r \cos \theta, y_0 + r \sin \theta)$$

in powers of r and $\theta - \theta_t$. It follows that if $z_r^{(k)}(0, \theta) = 0$, $0 \leqq \theta \leqq 2\pi$, $k = 1, 2, \cdots, 2n-1$,

$$\Delta_t z = z(r, \theta) - z(0, \theta_t) = (r, \theta - \theta_t)_{2n} + L_{2n},$$

where $(r, \theta - \theta_t)_{2n}$ denotes the sum of terms of degree $2n$ in r and $\theta - \theta_t$, and L_{2n} is finite near $(0, \theta_t)$. Consequently, in this ambiguous case, there are three possibilities:

1) *If* $(r, \theta - \theta_t)_{2n}$ *is a definite form for each value of* t, *then* $f(x_0, y_0)$ *is an extreme*; *a minimum or a maximum according as* $z_r^{(2n)}(0, \theta)$ *is positive or negative for* $\theta \neq \theta_t$, *respectively.*

2) *If* $(r, \theta - \theta_t)_{2n}$ *is an indefinite form for at least one value of* t, *then* $f(x_0, y_0)$ *is not an extreme.*

3) *If* $(r, \theta - \theta_t)_{2n}$ *is indefinite for no value of* t *but is semi-definite for at least one value of* t, *then we cannot say whether or not* $f(x_0, y_0)$ *is an extreme.*

The first coefficient $z_r^{(2n)}(0, \theta_t)$ in the form $(r, \theta - \theta_t)_{2n}$ is zero for each value of t. It is not difficult to show by direct differentiation that the *last two* coefficients are also zero for each value of t. Consequently, in the ambiguous case the resulting form $(r, \theta - \theta_t)_{2n}$ *always has its first and its last two terms zero.*

It follows that if $n = 1$, then

$$(r, \theta - \theta_t)_2 \equiv 0.$$

In this case it is necessary to proceed to the next term in $\Delta_t z$ to determine which of the three possibilities arises.

As a simple illustration, let us consider Peano's example

$$z = f(x, y) = y^2 - (p^2 + q^2)x^2 y + p^2 q^2 x^4.$$

The point (0, 0) is the only possible extreme for this function. Thus, we have

$$z(r, \theta) = r^2 \sin^2 \theta - r^3(p^2 + q^2) \sin \theta \cos^2 \theta + r^4 p^2 q^2 \cos^4 \theta$$

and

$$z_r^{(2)}(0, \theta) = 2 \sin^2 \theta,$$

which vanishes for $\theta_i = 0$ (not necessary to consider π since r is positive or negative) and we have Case III. However, we find that

$$\Delta_0 z = p^2q^2r^2 - (p^2 + q^2)r^3\theta + r^2\theta^2 + L_4,$$

and the form

$$(r, \theta - 0)_4 = r^2(p^2r - \theta)(q^2r - \theta)$$

is indefinite if $p \neq q$ and $f(0, 0)$ is not an extreme. If $p = q$, then this form is semi-definite and no conclusion can be drawn. However, it is easily seen directly that in this case $f(0, 0)$ is an improper minimum and the surface is tangent to the xy-plane along the parabola $y = p^2x^2$.

3. One-sided extremes. We shall derive necessary conditions and sufficient conditions in order that a function

$$z = f(x, y)$$

shall have a relative extreme at a point (x_0, y_0) on the boundary of the region R of admissible points. In the general case to be treated first, not both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ vanish at (x_0, y_0).

Let

$$C: \quad y = g(x)$$

represent the equation of the boundary of R near the point (x_0, y_0). The function

$$z = f[x, g(x)] \equiv Z(x)$$

has the derivative

$$Z^{(1)}(x) = f_x + f_y g^{(1)}(x) \tag{3}$$

near the point (x_0, y_0) provided $f(x, y)$ is of class C' in an extended region R' containing R in its interior and C is of class C' near (x_0, y_0). Throughout this paper R' denotes a region containing the admissible region R in its interior.

Now the equation

$$z_0 = f(x, y)$$

has the solution (x_0, y_0) at which we shall assume that $f_y(x_0, y_0) \neq 0$. Therefore, there exists a function

$$C_0: \quad y = g_0(x)$$

which satisfies this equation near (x_0, y_0) and has the same continuity class as $f(x, y)$ in R' near (x_0, y_0). Thus, we have the identity

$$z_0 = f[x, g_0(x)],$$

from which it follows that

$$0 = f_x + f_y g_0^{(1)}(x). \tag{4}$$

From equations (3) and (4), we have

$$Z^{(1)}(x_0) = f_y(x_0, y_0)[g^{(1)}(x_0) - g_0^{(1)}(x_0)].$$

THEOREM 6. *If $f(x, y)$ is of class C' in R' and $g(x)$ is of class C' near the point (x_0, y_0) on the boundary C at which $f(x, y)$ has a relative extreme with respect to the admissible region R, and if $f_y(x_0, y_0) \neq 0$, then*

$$g^{(1)}(x_0) = g_0^{(1)}(x_0).$$

It can also be shown that

$$Z^{(n)}(x_0) = f_y(x_0, y_0)[g^{(n)}(x_0) - g_0^{(n)}(x_0)]$$

by differentiating the relations (3) and (4) $n-1$ times and subtracting the resulting equations member by member, provided

$$g^{(k)}(x_0) = g_0^{(k)}(x_0), \qquad k = 1, 2, 3, \cdots, n-1.$$

THEOREM 7. *If $f(x, y)$ is of class C^{2n} in R' and $g(x)$ is of class C^{2n} near the point (x_0, y_0) on the boundary C at which $f_y(x_0, y_0) \neq 0$ and*

$$\begin{aligned} g^{(k)}(x_0) &= g_0^{(k)}(x_0), \qquad k = 1, 2, \cdots, 2n-1, \\ f_y(x_0, y_0)[g^{(2n)}(x_0) &- g_0^{(2n)}(x_0)] > 0, \end{aligned} \tag{5}$$

then $f(x_0, y_0)$ is a one-sided relative minimum

1) *with respect to the region $R[y \geqq g(x)]$ if $f_y(x_0, y_0) > 0$,*
2) *with respect to the region $R[y \leqq g(x)]$ if $f_y(x_0, y_0) < 0$.*

If the inequality in (5) is reversed, then $f(x_0, y_0)$ is a one-sided relative maximum

1) *with respect to the region $R[y \leqq g(x)]$ if $f_y(x_0, y_0) > 0$,*
2) *with respect to the region $R[y \geqq g(x)]$ if $f_y(x_0, y_0) < 0$.*

We shall consider the condition for a minimum where $f_y(x_0, y_0) > 0$, in which case the admissible region is $R[y \geqq g(x)]$. Then $f_y(x, y) > 0$ in an admissible neighborhood of (x_0, y_0) and $f(x, y)$ is increasing in y for each x in this neighborhood sufficiently near (x_0, y_0). But $f(x, y) = z_0$ at points of C_0 near (x_0, y_0). Therefore, $f(x, y) > z_0$ beyond C_0 along lines $x = x_k$ near (x_0, y_0). Also, since $f_y(x_0, y_0) > 0$, it follows that

$$g^{(2n)}(x_0) > g_0^{(2n)}(x_0)$$

and C_0 does not cross C at (x_0, y_0). Also, C is beyond C_0 along the lines $x = x_k$. Therefore, for all admissible points near (x_0, y_0) but distinct from it, $f(x, y) > z_0$ and $f(x_0, y_0) = z_0$. Consequently, $f(x_0, y_0)$ is a one-sided relative minimum. The other parts of the theorem can be proved in a similar manner.

THEOREM 8. *If $f(x, y)$ is of class C^{2n+1} in R' and $g(x)$ is of class C^{2n+1} near the point (x_0, y_0) on the boundary of R at which*

$$g^{(k)}(x_0) = g_0^{(k)}(x_0), \qquad k = 1, 2, \cdots, 2n,$$

$$f_y(x_0, y_0)[g^{(2n+1)}(x_0) - g_0^{(2n+1)}(x_0)] \neq 0,$$

then (x_0, y_0) is not an extreme of $f(x, y)$ with respect to the region R.

This follows from the fact that under these conditions the function $Z(x)$ does not have an extreme at $x = x_0$, that is, $f(x, y)$ does not have a *constrained* relative extreme at (x_0, y_0). In this connection it should be pointed out that Theorem 6 contains a necessary condition and Theorem 7 sufficient conditions in order that $f(x, y)$ have a constrained extreme at (x_0, y_0) along the curve C.

THEOREM 9. *If $f(x, y)$ is of class C' in R' and $g(x)$ is of class D' near a corner (x_0, y_0) at which $f_y(x_0, y_0) \neq 0$ and $f(x_0, y_0)$ is a one-sided relative minimum with respect to R, then*

$$f_y(x_0, y_0)[g^{(1)}(x_0+) - g_0^{(1)}(x_0)] \geqq 0, \qquad f_y(x_0, y_0)[g^{(1)}(x_0-) - g_0^{(1)}(x_0)] \leqq 0.$$

These inequalities are reversed if $f(x_0, y_0)$ is a one-sided relative maximum.

This theorem follows from an application of necessary conditions for a one-sided minimum of the function $Z(x)$ above to each branch of C at (x_0, y_0), at which the derivatives of $g(x)$ on the left and right are assumed to exist.*

THEOREM 10. *If $g(x)$ consists of two functions near (x_0, y_0), one of class C^N preceding and one of class C^n following the point (x_0, y_0); $f(x, y)$ is of class C^m in R', where m is the larger of N and n; and*

$$g^{(k)}(x_0+) = g_0^{(k)}(x_0), \qquad k = 1, 2, \cdots, n-1,$$

$$f_y(x_0, y_0)[g^{(n)}(x_0+) - g_0^{(n)}(x_0)] > 0,$$

$$g^{(s)}(x_0-) = g_0^{(s)}(x_0), \qquad s = 1, 2, \cdots, N-1, \tag{6}$$

$$f_y(x_0, y_0)[g^{(N)}(x_0-) - g_0^{(N)}(x_0)] \begin{cases} > 0, & N \text{ even}, \\ < 0, & N \text{ odd}, \end{cases}$$

then $f(x_0, y_0)$ is a one-sided relative minimum

1) *with respect to the region $R[y \geqq g(x)]$ if $f_y(x_0, y_0) > 0$,*
2) *with respect to the region $R[y \leqq g(x)]$ if $f_y(x_0, y_0) < 0$.*

If the inequalities in (6) are reversed, then $f(x_0, y_0)$ is a one-sided relative maximum

1) *with respect to the region $R[y \leqq g(x)]$ if $f_y(x_0, y_0) > 0$,*
2) *with respect to the region $R[y \geqq g(x)]$ if $f_y(x_0, y_0) < 0$.*

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* *Cf.* Mancill, One-sided Maxima and Minima, this MONTHLY, vol. 55, 1948, p. 313.

These results follow from an argument§ similar to that for Theorem 7, applied to each branch of C near (x_0, y_0).

This concludes the study of the general case of functions of two variables in which not both f_x and f_y are zero at (x_0, y_0), for if $f_y = 0$ and $f_x \neq 0$, then all the above conditions hold with the roles of x and y interchanged.

In the special case of one-sided extremes in which both f_x and f_y vanish at the point (x_0, y_0) on the boundary C, one can apply the results of the preceding section and, in general, determine whether (x_0, y_0) is a one-sided extreme of $f(x, y)$ with respect to the admissible region R except possibly in the ambiguous case.

4. One-sided extremes of functions of three or more variables. We shall consider the function

$$w = f(x, y, z)$$

for (x, y, z) in a closed region R and derive conditions analogous to theorems 6, 7, and 8 of the preceding section. The generalization of these conditions to functions of more than three variables is a mere routine involving only a problem of notation.

Let

$$S: \quad z = g(x, y)$$

represent the equation of the boundary surface of R near the point (x_0, y_0, z_0), denoted by P_0, on S. The function

$$w = f[x, y, g(x, y)] \equiv W(x, y)$$

has the derivative

$$W_x(x, y) = f_x + f_z g_x(x, y)$$

near the point P_0 in R provided $f(x, y, z)$ is of class C' in an extended region R' containing R in its interior and S is of class C' near P_0.

Now the equation

$$w_0 = f(x, y, z)$$

has the solution (x_0, y_0, z_0) at which we shall assume that $f_z(x_0, y_0, z_0) \neq 0$, since we are to consider the general case where not all the derivatives f_x, f_y, and f_z are zero. Therefore, there exists a function

$$S_0: \quad z = g_0(x, y)$$

which satisfies this equation near P_0 and has the same continuity class as $f(x, y, z)$ in R'. Thus, we have the identity

$$w_0 = f[x, y, g_0(x, y)]$$

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§ See also, Mancill, *loc. cit.*, p. 314.

from which it follows that

$$0 = f_z + f_z g_{0z}(x, y). \tag{8}$$

From equations (7) and (8), we have

$$W_z(x_0, y_0) = f_z(x_0, y_0, z_0)[g_z(x_0, y_0) - g_{0z}(x_0, y_0).]$$

In a similar way we can show that

$$W_y(x_0, y_0) = f_z(x_0, y_0, z_0)[g_y(x_0, y_0) - g_{0y}(x_0, y_0)],$$

and generally

$$\frac{\partial^n W(x_0, y_0)}{\partial x^h \partial y^k} = f_z(x_0, y_0, z_0)\left(\frac{\partial^n g(x_0, y_0)}{\partial x^h \partial y^k} - \frac{\partial^n g_0(x_0, y_0)}{\partial x^h \partial y^k}\right), \qquad h + k = n.$$

THEOREM 11. *If $f(x, y, z)$ is of class C' in R' and $g(x, y)$ is of class C' near P_0 on the boundary S at which $f(x, y, z)$ has a one-sided relative extreme with respect to R and $f_z(x_0, y_0, z_0) \neq 0$, then*

$$g_x(x_0, y_0) = g_{0x}(x_0, y_0), \qquad g_y(x_0, y_0) = g_{0y}(x_0, y_0).$$

THEOREM 12. *If $f(x, y, z)$ is of class C^{n+1} in R' and $g(x, y)$ is of class C^{n+1} near P_0 on the boundary S, where $f_z(x_0, y_0, z_0) \neq 0$; if*

$$\frac{\partial^m W(x_0, y_0)}{\partial x^h \partial y^k} = 0, \qquad h + k = m,\ m = 1, 2, \cdots, n - 1,$$

and if the form

$$\left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^n W(x, y) \tag{9}$$

at P_0 is positive definite in dx and dy, then $f(x_0, y_0, z_0)$ is a one-sided relative minimum

1) *with respect to the region $R[z \geqq g(x, y)]$ if $f_z(x_0, y_0, z_0) > 0$,*
2) *with respect to the region $R[z \leqq g(x, y)]$ if $f_z(x_0, y_0, z_0) < 0$.*

If the form (9) is negative definite, then $f(x_0, y_0, z_0)$ is a one-sided relative maximum

1) *with respect to the region $R[z \leqq g(x, y)]$ if $f_z(x_0, y_0, z_0) > 0$,*
2) *with respect to the region $R[z \geqq g(x, y)]$ if $f_z(x_0, y_0, z_0) < 0$.*

If the form (9) is indefinite, then $f(x_0, y_0, z_0)$ is not an extreme.

If the form (9) is semi-definite, then no conclusion can be drawn.

These conclusions can be proved similarly to those in Theorem 7.

It should be pointed out that theorems 11 and 12 contain necessary conditions and sufficient conditions, respectively, for $f(x_0, y_0, z_0)$ to be a constrained extreme of $f(x, y, z)$ on the surface S.

THE LENGTHS OF CURVES

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1. The problem stated. There can be no doubt that the first persons who interested themselves at all in the study of curves had very definitely in mind that a curve had a definite length. A curve appeared as a portion of a thin cord which might always be pulled out to its greatest extent, at which time the length would be evident. The question of how lengths were to be compared on different curves must have presented almost insoluble difficulties to reflective geometers, but to the uninitiated the matter was quite simple. We see in [1] that some 2500 years before Christ the Babylonians assumed that the circumference of a circle was three times its diameter; whether this was thought to be strictly accurate or to be a sufficiently correct approximation to the fundamental number of mathematics is not clear. We have no information as to the early Egyptian views on the matter but we know from [2] that some eighteen hundred years before the Christian era they had for π the excellent approximation 256/81. Euclid gives in XII [3] an excellent proof that circles are to one another as the squares of their radii, the proof being based on Eudoxus [3]. Euclid's proof was founded on the theorem that if from the greater of two magnitudes we subtract a quantity greater than or equal to one half itself, and continue to do this indefinitely, we shall eventually have a remainder which is less than the smaller of the two given magnitudes. The root idea is suggested by Eudoxus and his fundamental method of exhaustion.

We must turn to Archimedes who made the first great steps in the matter of finding the lengths of curves. In his *Sphere and Cylinder* we have the fundamental assumption:

A curve is concave in one direction if every line which connects two of its points has all points of the curve lying on one side of the line or on the line itself and none on the other side. It is then assumed (as axioms):

(A) Of all lines connecting two points the straight line is the shortest.

(B) Of all lines in the plane having the same extremities two are unequal when both are concave in the same direction, and that one of them included between the other and the straight line connecting the two points is the lesser.

(C) The length of a curve is a magnitude under the definition of Euclid X [1].

From these assumptions Archimedes proceeds to build up the measurement of circular arcs. We learn from *Sphere and Cylinder* that the perimeter of a circumscribed polygon is greater than that of a circle, and that we can circumscribe and inscribe two polygons so that the ratio of the circumscribed to the inscribed is less than any number greater than unity. In his book on the circle we learn [5] that its area is equal to that of a triangle one of whose lengths is equal to the circumference and the other to a radius. Finally he shows that π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$.

It is worth noting that this truly astounding mathematician in his treatise *On Spirals* studied totally different curves and developed a totally different technique for finding their lengths. Here a curve is looked on kinematically; it is traced by a point which moves by two different impetuses, and the length depends upon the relative strength of the two. This approach we shall meet later when we come to Roberval, to whom I shall return presently. It is worth noticing that Hippocrates of Chios had studied the transformation of other curves into combinations of circular arcs, but gave no indication of the metrical relations involved. The determination of the number π I leave entirely aside, referring to Cantor. A geometrical approach will be found in [6] and to the whole subject in [7]. The earliest infinite process leading to π was Vieta's

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}} \cdot \sqrt{\tfrac{1}{2} + \sqrt{\tfrac{1}{2}}} \cdot \sqrt{\tfrac{1}{2} + \sqrt{\tfrac{1}{2} + \sqrt{\tfrac{1}{2}}}} \cdots .$$

Let us return to Cavalieri [9] and Roberval, who lived at a time when the study of curves and their tangents was much in evidence. Roberval had very much in mind Archimedes' kinematical idea of resolving a motion into two parts, and taking the geometrical sum. A curve is traced by a moving point, the tangent anywhere is the line of instantaneous motion of that point. The real philosophical difficulty, to define just what is meant by instantaneous motion, was veiled in the future to bedevil those successors who occupied themselves with the foundations of the calculus. To the unspoiled eye of common sense there was no difficulty. Here is the *Axiome* or *principe d'invention*; "*La direction du mouvement d'un point qui décrit une ligne courbe est la touchante de la ligne courbe en chaque position de ce point là* [8]." This seems somewhat tautological; a tangent is a line which touches, but he goes on to explain: "*Par les proprietés spécifiques (qui vous seront données) examinez les divers mouvements qu'a le point à l'endroit ou vous voulez mener la touchante; de tous ces mouvements composez en un seul et tirez la ligne de direction du mouvement composé, vous aurez la touchante a la courbe.*"

The meaning is this. Determine two measurements which connect the moving point with two fixed elements. Determine the vector velocities of the change of the two; their vector sum will be the instantaneous velocity. The easiest example is the tangent to the parabola. Here the distances to the focus and directrix are equal, hence the tangent to the curve makes equal angles with the focal radius and with a parallel to the axis.

2. Actual Lengths. The question of actually rectifying a curve with a continuously moving tangent occupied the attention of a number of contemporaneous geometers of the seventeenth century. There is a good deal to be said for the thesis that the first writer actually to do this was Torricelli [10]. He claims the discovery of the curve which we should write $\rho = e^{i\phi}$ and of which the subtangent is always equal in length to the preceding arch of the curve. There is some

doubt about the originality of this discovery, for it is usually ascribed to Descartes who communicated it to Mersenne and who in turn may have communicated it to Torricelli. I think, however, that on the whole the Italian should receive the credit for the discovery. In any case he did not long enjoy any primacy, for in 1657 a not very distinguished English mathematician (*cf.* Wallis in [11]), William Neil, showed how you could rectify a semi-cubical parabola by comparing its length with another known curve. In 1658 a very easy solution to the problem of rectifying the cycloid was published by Christopher Wren and was published by Wallis [11, p. 533]. This came presently to the notice of Fermat who in 1660 published [12]. He acknowledged Wren's rectification of the cycloid, and his editors, Tannery and Henry, acknowledge that he may have been familiar with other writers on the same subject, but in his own method there was certainly a novel and interesting element. Archimedes had laid down the method of exhaustion that the length of a curve differed by less than any assigned quantity from that of either an inscribed or a circumscribed curve; Fermat's idea was to use two circumscribed curves.

Let us take along the arc of a curve (which is concave downward) tangents at points whose abscissas are evenly spaced. Consider first the portion of each tangent from its point of contact to the intersection with the ordinate at the preceding point, the ordinates being supposed to increase. The length of this backward section of the tangent is less than the chord and so less than the section of the arc. On the other hand the length of the section of each tangent from its point of contact to the intersection with the ordinate of the following point is greater than the corresponding section of arc. Adding the backward sections of the tangents together and then the forward sections, we have two circumscribed "polygons" whose difference can be made as small as we please. We are then back to Archimedes. Fermat takes the curve $y^3 = kx^2$ and shows that its rectification can be thrown back on $y^2 = lx$, and this was done by Archimedes.

Another writer who should be considered in this connection was Barrow whose studies were elaborately discussed in [13]. He reverts to Archimedes' study of the motion of a point by considering this as composed of two separate motions, and finding their resultant. In the first examples a straight line moves parallel to itself, let us say uniformly, while a point moves down that line according to some known law. We study the motion of that point. On p. 120 of [13] he discusses briefly a method of drawing tangents "by calculation frequently used by us. Although I hardly know after so many well known and well worn methods of the kind above whether there is any advantage in doing so." The method so introduced was the formula of de Sluse [14] who as early as 1652 had some sort of a method for drawing tangents. The subtangent to $f(x,y) = 0$ may be written $a = -yf_y/f_x$.

Newton himself made no contribution to our general question, but he states as a well known theorem that the fluxion of an element of an arc is the square root of the sum of the squares of the fluxions of the two ends. This tacitly assumes that an element has a determinate fluxion.

3. The fundamental theorem. Let us return again to our fundamental question which is carefully studied in [15]. The first person to take this up seems to have been Dirksen [16]. He points out that a mathematical statement is either a definition, or an axiom or a theorem. Which category should we assign to Archimedes' statements about lengths? His statement, (B), about unequal lengths cannot be counted as an axiom, for we have no immediate way of comparing lengths on lines of different curvature. If we treat it as a theorem, a proof would involve a discussion of the infinitesimal domain that would have surely been beyond the reach of Archimedes' contemporaries and might call for the use of essentially equivalent statements. There remains only the possibility of its being a definition, but it must also be put into a more usable form.

Take a section of a curve AB which is concave in one sense and is cut by n equally spaced parallel lines in n points, each connected with the preceding and the succeeding by straight lines. The limit of the sum of these line segments as n increases indefinitely shall be defined as the length of the arc AB.

Dirksen proceeds to show that the length of the sum of two such arcs is the sum of their lengths. This seems to me to have been a real attempt to put the whole subject on a sound basis, but was still lacking in detail. The introduction of the equally spaced lines was unfortunate, and we do not know that the length of an arc is independent of the method of dividing it up. There was no direct method suggested for comparing different arcs. The suggestion was put forth once or twice that we might define as the length of an arc the lower limit of the lengths of circumscribed polygons as the number of sections increased indefinitely, but this ran straight into the difficulty that since there were an infinite number of circumscribed polygons of different lengths, how could we be sure that the lower limit existed? True, none of these could be less than the straight line, but that did not solve the problem completely. An ambitious attempt appeared in [17] but this was better done in [18].

Following Rouché and Comberouse, let us suppose that we have an arc of a curve concave in one sense. We take any number of small sections which together make up the arc, and inscribe and circumscribe polygons, the circumscribed being tangent at the vertices of the inscribed. We define as the length of the arc the limit which the lengths of the inscribed polygons approach as the length of each chord approaches zero. It remains to prove that this length is independent of the method of subdivision.

Let L be the length which is obtained from a particular method of subdivision. Also let the circumscribed vertices be $M'N'I' \cdots$, the inscribed being $ABC, \cdots$, while $MNI \cdots$ are the projections of the circumscribed vertices on the inscribed polygon.

Consider the fraction $AM'+M'B+BN'+ \cdots /AM+MB+BN+ \cdots$. The ratio of each term of the numerator to the corresponding term of the denominator, say AM'/AM, is greater than 1, but tends to 1 as the lengths of the

subarcs tend to zero. Hence the limit of the fraction above will be unity. Let p be the length of one of the inscribed polygons, and P that of a circumscribed polygon. Then

$$p \leqq L \leqq P; \qquad 1 \leqq \frac{L}{p} \leqq \frac{P}{p}.$$

Since $\lim P/p = 1$, it follows that $\lim L/p = 1$, or $L = \lim p$ regardless of the method of subdivision.

We have successfully avoided the difficulty of Dirksen of using a particular division, but this proof is certainly more dependent on intuition.

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NORMAL AND DIAGONALIZABLE MATRICES

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Normal matrices and matrices which can be diagonalized by similarity transformation are closely connected. While this is to be expected, since the former is a subset of the latter, the connection is not usually pointed out. It might be interesting then to consider various criteria for normal matrices and to note the corresponding analogues for diagonalizable matrices. The relation between the two classes of matrices is given by the following theorem.

THEOREM 1. *The matrix A is diagonalizable if and only if there exists a positive definite hermitian matrix H such that $H^{-1}AH=N$ is normal.*

The sufficiency is clear since a normal matrix may be unitarily diagonalized.

For the necessity suppose A may be diagonalized, say $P^{-1}AP=D$. By a theorem of Autonne [1], $P=HU$ where H is positive definite hermitian and U is unitary. Hence $H^{-1}AH=UDU^*=N$ is normal.

Thus in general any necessary and sufficient condition for a matrix A to be normal becomes a necessary and sufficient condition for A to be diagonalizable when applied to $H^{-1}AH$ for a positive definite hermitian matrix H.

The Toeplitz condition [7] on unitary diagonalization of a normal matrix may be stated in the equivalent form: A is normal if and only if it has n linearly independent orthogonal characteristic vectors. For A to be diagonalizable we merely require that it have n linearly independent characteristic vectors since if $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent there always exists a positive definite hermitian matrix H such that $H^{-1}\phi_1, H^{-1}\phi_2, \cdots, H^{-1}\phi_n$ are orthogonal.

Williamson [8] proved that A is normal if and only if A^* is a polynomial in A. The analogue is: A is diagonalizable if and only if $H^{-1}A^*H$ is a polynomial in A for H positive definite hermitian.

For if A is diagonalizable $H^{-1}AH=N$ is normal and $N^*=f(N)=H^{-1}f(A)H$. Hence $HA^*H^{-1}=H^{-1}f(A)H$ and $H^2A^*H^{-2}=f(A)$.

Conversely if $H^{-1}A^*H=f(A)$ then since H is positive definite hermitian it has a unique positive definite hermitian square root H_1. Then $H_1^{-1}A^*H_1=H_1f(A)H_1^{-1}=f(H_1AH_1^{-1})$, and so $H_1AH_1^{-1}$ is normal.

Since the analogues for the remaining criteria are obtained easily, we shall no longer explicitly state them.

THEOREM 2 (Parker [6]). *Let the characteristic roots of A be $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then A is normal if and only if the characteristic roots of A^*A (or of AA^*) are $\lambda_1\bar{\lambda}_1, \cdots, \lambda_n\bar{\lambda}_n$.*

The necessity is obvious.

For the sufficiency let A be unitarily triangularized, say $U^*AU=D=(d_{ij})$. Then $\operatorname{tr}(D^*D)=\operatorname{tr}(U^*A^*AU)=\sum_{i=1}^n\lambda_i\bar{\lambda}_i+\sum_{i\neq j}d_{ij}\bar{d}_{ij}=\operatorname{tr}(A^*A)=\sum_{i=1}^n\lambda_i\bar{\lambda}_i$. Hence $\sum_{i\neq j}d_{ij}\bar{d}_{ij}=0$ and D is diagonal.

Parker also proves in this paper that A is normal if and only if the roots of

$A+A^*$ (or of A^*+A) are $\lambda_1+\bar{\lambda}_1, \cdots, \lambda_n+\bar{\lambda}_n$. This theorem is a consequence of a theorem of Goldhaber [3]. For it is shown there that any two matrices A_1, A_2 with the property that the roots of A_1+A_2 are the sum of the roots of A_1 and A_2 satisfy Frobenius' Theorem and hence (McCoy [5]) may be simultaneously unitarily triangularized. But the assertion that A and A^* may be simultaneously unitarily triangularized implies that A may be unitarily diagonalized, and hence is normal.

This same principle will yield other criteria for normal matrices. For instance it is well known that any matrix A may be uniquely expressed in the form $A=H+K$ where H is hermitian and K is skew-hermitian, and that A is normal if and only if $HK=KH$. Application of the above-mentioned theorem of McCoy gives the result that A is normal if and only if H, K satisfy Frobenius' Theorem, or if and only if any two of A, H, K may be simultaneously unitarily triangularized (and hence diagonalized).

THEOREM 3. *A necessary and sufficient condition for A to be normal is that any characteristic vector of A be also a characteristic vector of A^*.*

The necessity follows from the fact that A^* is a polynomial in A.

For the sufficiency suppose $A\xi=\lambda\xi$ implies $A^*\xi=\beta\xi$. Then $\xi^*A^*\xi=\bar{\lambda}=\beta$. Hence if $U=(\xi, *, \cdots, *)$ is unitary,

$$U^*AU=\begin{bmatrix}\lambda & 0\\ 0 & A_1\end{bmatrix} \quad \text{and} \quad U^*A^*U=\begin{bmatrix}\bar{\lambda} & 0\\ 0 & A_1^*\end{bmatrix}.$$

The theorem is obviously true for $n=1$. Assume it is true for $n-1$. Then, since A_1 is of order $n-1$ and any characteristic vector of A_1 is also one of A_1^*, A_1 may be unitarily diagonalized by the induction hypothesis. So therefore may A, and hence A is normal.

COROLLARY 1. *A necessary and sufficient condition for A to be normal is that if ξ is any characteristic column vector of A then ξ^* is a characteristic row vector.*

COROLLARY 2. *A necessary and sufficient condition for A to be normal is that any space invariant under A also be invariant under A^*.*

The necessity follows from the fact that A^* is a polynomial in A.

The sufficiency follows from the theorem.

COROLLARY 3 (Halmos [4], page 132). *A necessary and sufficient condition for A to be normal is that if S is any space invariant under A then S', the orthogonal complement of S, is also invariant under A.*

Since, for any matrix A, S invariant under A implies S' invariant under A^*, this corollary is equivalent to Corollary 2.

Note. The writer was unable to obtain Drazin's paper [2] until some months after completion of the present paper. Despite the fact that titles and topics are nearly identical, there is very little overlap in content, and almost none in

method.

His Theorem 1 (i) is an immediate consequence of the Jordan normal form of A. It is equivalent to each of the following statements: For every characteristic root α of the matrix A,

(1) nullity $(A-\alpha I)^k$ = nullity $(A-\alpha I)$ for all positive integers k,

(2) The Weyr characteristic of A relative to α is $r, 0, \cdots, 0$ where r is the nullity of $(A-\alpha I)$,

(3) The Segré characteristic of A relative to α is $1, 1, \cdots, 1$ to r terms.

Statements (i), (1), (2), and (3) are all corollaries to the theorem of Weierstrass (1861) that a matrix is diagonalizable if and only if it has only simple elementary divisors.

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MATHEMATICAL NOTES

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SOME NECESSARY CONDITIONS FOR CONVERGENCE OF INFINITE SERIES AND IMPROPER INTEGRALS

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If the terms of a convergent infinite series, $\sum a_n$, are non-increasing, then $\lim_{n\to\infty} na_n$ is zero. Prompted by this fact and the logarithmic comparison tests, we present a chain of necessary conditions for convergence. Analogous tests are first developed for improper integrals, and the results for series are then easily obtained. Throughout the discussion we assume, for convenience, that $a>0$

and that a and m are large enough so that any required logarithms are positive.

Our starting point is the known result for integrals. A simple proof may be modeled after the one for series as given in K. Knopp, *Theory and Application of Infinite Series*, p. 124.

THEOREM. *If $f(x)$ is non-increasing and $\int_a^\infty f(x)dx$ is convergent, then $\lim_{x\to\infty} f(x)x$ is zero.*

In general we have the following result, where $\log^{(k)} x$ stands for the logarithm iterated k times.

THEOREM 1. *If $f(x)\ x \log x \log\log x \cdots \log^{(k-1)}x$ is non-increasing and $\int_a^\infty f(x)dx$ is convergent, then*

$$\lim_{x\to\infty} f(x)x \log x \log\log x \cdots \log^{(k-1)} x \log^{(k)} x \text{ is zero.}$$

Proof. For the case that $k=1$, the substitution, $x=e^y$, yields the convergent integral $\int_{\log a}^\infty f(e^y)e^y dy$. Since $f(x)x$ is non-increasing by hypothesis, $f(e^y)e^y$ cannot increase as y increases. By the previous theorem $\lim_{y\to\infty} f(e^y)e^y y$ is zero. Therefore $\lim_{x\to\infty} f(x)x \log x$ is zero.

The general case is treated by making the substitution, $x=\exp^{(k)}y$.

We now prove the corresponding theorem for series.

THEOREM 2. *If $\sum_{n=m}^\infty a_n$ is a convergent infinite series such that $a_n n \log n \cdots \log^{(k-1)}n$ is non-increasing, then $\lim_{n\to\infty} a_n n \log n \cdots \log^{(k-1)}n \log^{(k)}n$ is zero.*

Proof. Theorem 2 will follow from an application of Theorem 1 to the function $f(x)$ defined for $n\leqq x<n+1 (n=m,\ m+1,\ \cdots)$ by the formula $f(x)=a_n n \log n \cdots \log^{(k-1)}n/(x \log x \cdots \log^{(k-1)}x)$. The second hypothesis of Theorem 2 implies that $f(x)$ has the first property required in Theorem 1. From the hypotheses of Theorem 2 it is easy to show that $a_n\geqq 0$, whence $0\leqq f(x)\leqq a_n$ for $n\leqq x<n+1 (n=m,\ m+1,\ \cdots)$. Thus $\int_m^\infty f(x)dx$ is dominated by $\sum_{n=m}^\infty a_n$, $f(x)$ meets the second requirement of Theorem 1, and Theorem 2 is proved.

GENERALIZATION OF CLAIRAUT'S DIFFERENTIAL EQUATION AND THE ANALOGOUS DIFFERENCE EQUATION

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In a recent note (this MONTHLY, vol. 59, page 100) W. H. Witty generalizes Clairaut's equation to

$$y - xy' + \frac{x^2y''}{2!} + \cdots + \frac{(-1)^{n-1}x^{n-1}y^{(n-1)}}{(n-1)!} + f(y^{(n)}) = 0$$

and solves it. It is the purpose of this note to point out a still wider generalization which is given with references, as a problem in Goursat-Hedrick, vol. II, part II, page 44, and also its analogue in difference equations.

The problem is to integrate

$$F\left(y'',\ y' - xy'',\ y - xy' + \frac{x^2y''}{2}\right) = 0$$

and it is noted that there exist equations of analogous form and of any order.

Consider the equation $F(z_0, z_1, \cdots, z_{n-1}) = 0$ where

$$z_r = y^{(r)} - xy^{(r+1)} + \frac{x^2y^{(r+2)}}{2!} + \cdots + \frac{(-1)^{n-r-1}x^{n-r-1}y^{(n-1)}}{(n-r-1)!}$$

$$= \sum_{s=0}^{n-r-1} \frac{(-1)^s x^s y^{(r+s)}}{s!} \quad \text{where} \quad y^{(r)} = \frac{d^r y}{dx^r}.$$

Thus

$$\begin{aligned} z_0 &= y - xy' + \frac{x^2y''}{2!} + \cdots + \frac{(-1)^{n-1}x^{n-1}y^{(n-1)}}{(n-1)!} \\ &\vdots \\ z_{n-2} &= y^{(n-2)} - xy^{(n-1)} \\ z_{n-1} &= y^{(n-1)}. \end{aligned}$$

It follows by induction that

$$\frac{dz_r}{dx} = y^{(n)}\phi_r(x) \tag{1}$$

where $\phi_r(x)$ are some functions of x. This suggests that a solution $y(x)$ of $F(z_0, \cdots, z_{n-1}) = 0$ may have $y^{(n)}(x) = 0$, or that it may be of the form:

$$y(x) = a_0 + a_1x + \frac{a_2x^2}{2} + \cdots + \frac{a_{n-1}x^{n-1}}{(n-1)!} = \sum_0^{n-1} \frac{a_r x^r}{r!}, \tag{2}$$

where the a's are yet to be determined. In this case the z_r are constants because of (1); and these constants must satisfy $F(z_0, \cdots, z_{n-1}) = 0$. From the definition of z_r, $z_r(0) = y^r(0) = a_r$. Hence $z_r = a_r$, and so $F(a_0, \cdots, a_{n-1})$ must be zero. Therefore the function (2) is a solution of $F(z_0, \cdots, z_{n-1}) = 0$ whenever $F(a_0, \cdots, a_{n-1}) = 0$. As this solution has $n-1$ arbitrary constants, it is the general solution.

The analogue of the generalized differential equation as a difference equation is

$$F(z_0, z_1, z_2, \cdots, z_{n-1}) = 0$$

where

$$z_0 = u_x - x\Delta u_x + \frac{x(x+1)\Delta^2 u_x}{2!} + \cdots + \frac{(-1)^{n-1}x(x+1)\cdots(x+n-2)\Delta^{n-1}u_x}{(n-1)!}$$

$$z_1 = \Delta u_x - x\Delta^2 u_x + \frac{x(x+1)\Delta^3 u_x}{2!} + \cdots + \frac{(-1)^{n-1}x(x+1)\cdots(x+n-3)\Delta^{n-1}u_x}{(n-2)!}$$

$$\vdots$$

$$z_{n-2} = \Delta^{n-2}u_x - x\Delta^{n-1}u_x$$

$$z_{n-1} = \Delta^{n-1}u_x.$$

It follows by induction that

$$\Delta z_r = \phi_r(x)\Delta^n u_x.$$

As before, this suggests a solution of the form:

$$(3) \qquad u_x = a_0 + a_1 x + a_2\frac{x(x-1)}{2!} + \cdots + a_{n-1}\frac{x(x-1)\cdots(x-n+1)}{(n-1)!} = \sum_0^{n-1} a_r\binom{x}{r}.$$

Also z_r are constants; $z_r(0) = \Delta^{(r)}u_0 = a_r$. So $z_r = a_r$, and $F(a_0, \cdots, a_{n-1}) = 0$. Therefore (3) is a solution of $F(z_0, \cdots, z_{n-1})$ whenever $F(a_0, \cdots, a_{n-1}) = 0$.

A MATRIC PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA FOR REAL QUATERNIONS

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In her doctoral thesis [7], Louise A. Wolf showed how one may discuss the theory of matrices whose elements are in the ring Q of real quaternions.* She made essential use of the isomorphism of Q_n onto a ring of matrices with complex elements which is accomplished by replacing each real quaternion by its matric form. (See, for example, [3, p. 60].) We observe here that a very minor consequence of the results of Wolf, combined with a general lemma of Hamilton-Cayley type (which applies in an arbitrary ring with unit) will yield a very simple proof of a† "fundamental theorem of algebra" for real quaternions.

Let R be a ring with unit 1 and let $f(x)$ be a non-constant monic polynomial with coefficients in R. We define $C_L(f)$, the *left* companion matrix of $f(x)$, as does MacDuffee [5, pp. 81–82] for the case in which R is a field. For example, if $f(x) = x^2 + a_1x + a_0$, then

* N. Jacobson [2, pp. 50–51] derived the results of Wolf as a special case of a more general theory of non-commutative principal ideal domains. Very recently, independent work by H. C. Lee [4] and by J. L. Brenner [1] on this topic has appeared.

† See our concluding remark.

$$C_L(f) = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix}.$$

In this special case a simple computation shows that

$$f_L(C_L(f)) = (C_L(f))^2 + (C_L(f))a_1 + a_0 I = 0.$$

By means of an inductive proof, which is straightforward and will be omitted, we have the following theorem.

THEOREM 1. *If R is a ring with unit element and $f(x)$ is a non-constant monic polynomial with coefficients in R, then $f_L(C_L(f)) = 0$.*

Now let $R = Q$ and let $f(x)$ be a non-constant monic polynomial with coefficients in Q. The fact that there is a nonzero vector v of real quaternions such that $vC_L(f) = qv$, for some q in Q, is a consequence of the results of Wolf and is very easy to prove directly using her isomorphism. Since $vf_L(C_L(f)) = 0$, we easily find that $f_L(t^{-1}qt) = 0$ for each nonzero component t of v. If $g(x)$ is a non-constant polynomial with coefficients in Q and with leading coefficient a_n, then $f(x) = g(x)a_n^{-1}$ is monic and has a left-hand root r in Q. Since $g(x) = f(x)a_n$, r is also a left-hand root of $g(x)$.

THEOREM 2. *Every non-constant polynomial with real quaternion coefficients has a left-hand root which is a real quaternion.*

Concluding remark. We have been unable to find a correspondingly simple proof of the more general form of the "fundamental theorem of algebra" for real quaternions which is due to I. Niven and S. Eilenberg [6].

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UNIQUE SEGMENTS IN METRIC SPACES

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Let M be a metric space, with the distance from x to y denoted by (xy). An ordered set of points $\langle x, p_1, p_2, \cdots, p_n, y\rangle$ is said to be linear if $(xp_1) + \cdots + (p_n y) = (xy)$. Let $[x, y]$, the linear join of x and y, be the set of all p such that $\langle x, p, y\rangle$ is linear. M is said to be *convex* if $[x, y]$ always contains a point other

than x and y. If M is complete and convex, $[x, y]$ is a continuum connecting x and y. M is said to have unique segments if $[x, y]$ is always a single arc and is therefore congruent to a Euclidean line segment of length (xy). Several conditions are known which are equivalent to this condition. The following is typical. In all that follows, M is assumed complete and convex.

THEOREM 1. *Let the linear joins of M obey the following condition. Given x, y, and points p and q in $[x, y]$, either $[x, p]$ intersects $[q, y]$ or $[p, y]$ intersects $[x, q]$. Then, M has unique segments.*

For, if $u \epsilon [x, p] \cap [q, y]$, then $(xu)+(up)=(xp)$, and $(xu)+(up)+(py)=(xy)$. Hence $(up)+(py)=(uy)$ so that $(xy)=(xq)+(qu)+(uy)=(xq)+(qu)+(up)+(py)$ and $\langle x, q, u, p, y\rangle$ is linear.

The purpose of this note is to show that a quadratic metric inequality also implies unique segments. Let us first recall that M is said to have the n-point property if every set of n points of M is congruent with a set of n points in a suitable Euclidean space [1]. Thus, the 3-point property is nothing more than the triangle law. Wilson [2] proved that if M has the 4-point property, M itself is congruent with a subset of a Euclidean space (possibly infinite dimensional), and Blumenthal strengthened this by requiring this hypothesis only for quadruples containing a linear triple [1]. In particular the 4-point property certainly implies that M has unique segments. Our condition will also be a condition on the mutual distances of 4-points which hold for the Euclidean spaces, but which is not as strong as the 4-point property. We first establish a general inequality for the mutual distances of four points in any metric space.

LEMMA. *For any four points x, y, p, q in M,*

$$(xy)^2 + (pq)^2 \leq (xp)^2 + (yq)^2 + 2(yp)[(xy) + (yq)]. \tag{1}$$

We have $|(py)-(xy)| \leq (px)$, and $(pq) \leq (py)+(yq)$ so that $[(py)-(xy)]^2 + (pq)^2 \leq (px)^2 + [(py)+(yq)]^2$ from which (1) immediately comes.

To obtain our 4-point condition on M, we strengthen (1) in a natural way by replacing the term $(xy)+(yq)$ by the smaller number (xq).

DEFINITION. *M has property Q if, for any x, y, p, q,*

$$(xy)^2 + (pq)^2 \leq (xp)^2 + (yq)^2 + 2(xq)(yp). \tag{2}$$

THEOREM 2. *If M has property Q, it has unique segments.*

Let p and q belong to $[x, y]$, with $(xp)=(xq)=a$ and set $(xy)=L$. Then, by (2)

$$L^2 + (pq)^2 \leq a^2 + (L-a)^2 + 2a(L-a)$$

from which $(pq)=0$ and $p=q$.

Property Q is not vacuous, since it obtains in all Euclidean spaces.*

THEOREM 3. *If M is a Euclidean space, (2) holds.*

In terms of the inner product in M, we have

$$\begin{aligned}(xy)^2 + (pq)^2 - (xp)^2 - (yp)^2 &= (x-y)\cdot(x-y) + (p-q)(p-q) - (x-p)(x-p) - (y-q)(y-q)\\ &= 2(x\cdot p) + 2(y\cdot q) - 2(x\cdot y) - 2(p\cdot q)\\ &= 2(y-p)(q-x)\\ &\le 2\|y-p\|\,\|q-x\| = 2(yp)(xq).\end{aligned}$$

It is easy to see that in E^n, equality in (2) forces the four points to be co-planar.

Property Q holds also in some spaces other than the Euclidean spaces. Consider for example the tripod space M formed by the positive Y axis, and the whole X axis, metrized by the geodesic metric in M. For some spaces, the inequality (2) may hold locally, although it fails in the large. For these, we will then have unique segments in the small. This is the case for a 1-sphere or 2-sphere. It would be of interest to know if property Q imposes any further condition on M, such as curvature restrictions, and what other spaces obey it.

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ON THE SKEW QUADRILATERAL†

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1. Two theorems. In this section we establish two theorems concerning the skew quadrilateral.

THEOREM 1. *A skew quadrilateral has its opposite sides equal in pairs if and only if it has its opposite angles equal in pairs.*

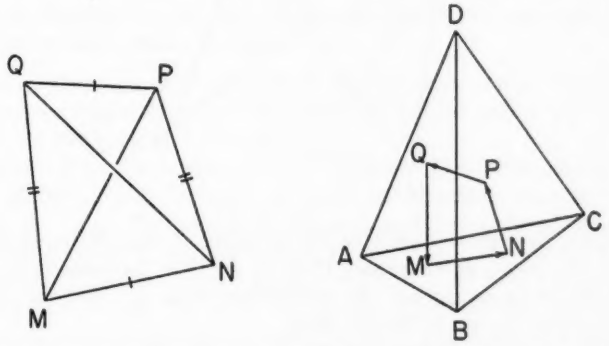
In fact, if the opposite sides MN and PQ of a skew quadrilateral $MNPQ$ are equal, as well as the opposite sides NP and QM, then triangles MNP and PQM are congruent and triangles MNQ and NPQ are congruent, and we have the angular equalities

$$(1) \qquad (MN, MQ) = (PQ, PN), \qquad (NP, NM) = (QM, QP).$$

* The referee has pointed out that the four-point inequality (2) has been used before in a different connection. (See L. Blumenthal, The norm of a configuration, Bull. Amer. Math Soc., vol. 34, 1928, pp. 726–730.)

† Translated from the French by Howard Eves.

Conversely, let us be given a skew quadrilateral $MNPQ$ for which the angular equalities (1) hold. Draw non-concurrent planes perpendicular to the sides QM, MN, NP, PQ in such a manner that the vectors $\overrightarrow{QM}$, $\overrightarrow{MN}$, $\overrightarrow{NP}$, $\overrightarrow{PQ}$ either all have the sense of the interior normals to the faces ABC, BCD, CDA, DAB



of the tetrahedron T determined by these planes or all have the sense of the exterior normals to these faces. Then the opposite dihedral angles BC and AD of T are equal, as well as the opposite dihedral angles CD and AB. Therefore the trihedral angles A-BCD and C-DAB have the three dihedral angles of one equal to the three dihedral angles of the other, which implies the angular equalities

$$(AB, AC) = (CD, CA), \qquad (AC, AD) = (CA, CB),$$

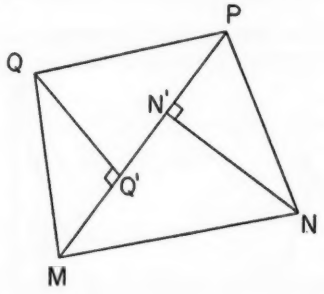
and, consequently, the congruency of triangles ABC and CDA. Now, since by a known theorem the lengths of the sides QM and NP of the quadrilateral are proportional to the areas of the faces ABC and CDA, we have $QM = NP$. Similarly, $MN = PQ$.

THEOREM 2. *A skew quadrilateral has its opposite sides equal in pairs if and only if the perpendiculars erected at the vertices to the planes of the two sides meeting at these vertices constitute a hyperbolic group of lines.*

In fact, for the perpendiculars m, n, p, q at the vertices M, N, P, Q to the planes (MN, MQ), (NP, NM), (PQ, PN), (QM, QP) to be hyperbolic, it is necessary and sufficient that the orthogonal projections on the plane of a face of the tetrahedron $MNPQ$ of the perpendiculars to the other three faces be concurrent lines, and that this property hold for three of the faces. Now the projections of m and p on the plane MNP pass through the point N_1 diametrically opposite to N on the circumference MNP, and the projection of q on the plane MNP coincides with the trace on that plane of the plane drawn perpendicular to MP through the altitude QQ' of triangle MPQ. If

$$MN = PQ, \qquad NP = QM, \tag{2}$$

then the feet N' and Q' of the altitudes NN' and QQ' of triangles MNP and



MPQ are symmetric with respect to the midpoint of side MP, and the projection of q on the plane MNP passes through N_1. We may establish analogous conclusions for the projections of the perpendiculars m, n, p and the perpendiculars m, n, q on the planes MPQ and NPQ.

Conversely, if the projections of the perpendiculars m, p, q on the plane MNP are concurrent in a point N_1, then N_1 is diametrically opposite vertex N on the circumference MNP, and equations (2) follow from

$$(QM)^2 - (QP)^2 = (NP)^2 - (NM)^2,$$
$$(MQ)^2 - (MN)^2 = (PN)^2 - (PQ)^2,$$

by the addition and subtraction of corresponding members.

2. Application to the tetrahedron. Let us designate by M and P, N and Q, the opposite angles of a skew quadrilateral $MNPQ$, and by M' and P', N' and Q', the supplements of these angles. If $M=P$ and $N=Q$ we also have

$$M + P' = P + M' = 180^\circ = N + Q' = Q + N'.$$

On the other hand we can never, for a skew quadrilateral $MNPQ$, have either

$$M + P = 180^\circ = N + Q \quad \text{or} \quad M' + P' = 180^\circ = N' + Q'.$$

To see this latter fact we note that the diagonal NQ of the quadrilateral determines the triangles MNQ and NPQ, the sum of the angles of which is equal to 360°. Since any face angle of a trihedral angle is less than the sum of the other two face angles, we have, first of all, the angular relations

$$Q < (QM, QN) + (QN, QP), \qquad N < (NQ, NM) + (NP, NQ),$$

whence

$$Q + N < 180^\circ - M + 180^\circ - P,$$

and, finally,

$$M + N + P + Q < 360°.$$

We may now state

THEOREM 3. *A tetrahedron cannot have two pairs of opposite dihedral angles supplementary* [1], *nor two pairs of opposite exterior dihedral angles supplementary, but it can have two or three pairs of an interior dihedral angle and an opposite exterior dihedral angle supplementary.*

It is the last configuration (in which case the tetrahedron is isosceles) that should have been considered in the announcement of the author's problem E 617 [1944, 231 and 1945, 160].

Translator's Note. The statement of E 617 is as follows: "From a given tetrahedron we derive another by taking as vertices the points of contact of the insphere with the faces. Show that the dihedral angles at pairs of opposite edges of the first tetrahedron are supplementary if, and only if, the second tetrahedron is trirectangular." The statement of this problem is at fault; to correct it we should replace the words "the insphere" by the words "an escribed sphere" and the word "supplementary" by the word "equal." The error in Kelly's "proof" of the incorrectly stated problem [1945, 160] lies in the fact that the center of the circumsphere of a trirectangular tetrahedron T lies *outside* T, and consequently this circumsphere cannot be the inscribed sphere, but must be an escribed sphere, of the tangential tetrahedron T' of T.

Reference

1. A. Marmion, *Mathesis*, t. 58, p. 103, question 3412.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

THE MATHEMATICAL TRAINING OF ENGINEERS*

RALPH HULL, Purdue University

The Mathematical Association of America has recently introduced into its programs for the annual summer meetings a series of symposia on topics related to the teaching of mathematics. The first symposium was held in September of last year at the University of Minnesota on the question: *What should we teach our undergraduate majors in mathematics, with special reference to their subsequent employment in industry?* The second symposium was held on September 2, 1952, at Michigan State College on the subject: *The mathematical training of engineers*, with Professor Ralph Hull, Head of the Purdue Department of Mathematics, acting as chairman. Three invited speakers presented prepared talks of about twenty-five minutes each on three different aspects of the subject. The talks were followed by an extended and lively discussion.

The first and third speakers were Professor E. B. Allen of the Department of Mathematics, Rensselaer Polytechnic Institute, who spoke on *Mathematics and the engineering curriculum*, and Professor Elizabeth S. Sokolnikoff, of the Department of Mathematics, University of Wisconsin, who spoke on *Problems of mathematicians who teach engineers*. These speakers emphasized the fact that the mathematical training of engineers is a joint responsibility of the faculty of engineering and of the department of mathematics, which may be in some other college of the university. The training can only be effective as a result of close cooperation between the Schools of Engineering and Departments of Mathematics. Such cooperation is generally found in our universities and present conditions are favorable to still closer cooperation. For example, more and more staff members of schools of engineering have had advanced mathematical training; recent textbooks in engineering show a tendency to draw more frequently on mathematical ideas and consequently more of the mathematics which has traditionally been presented in courses on the calculus and differential equations is actually used in engineering courses. On the other hand, many mathematicians have worked directly with engineers in industry or in government research laboratories since the beginning of World War II and consequently are more familiar with the mathematical needs of engineers.

Various ways were suggested of developing still closer relationships between departments of mathematics and the schools of engineering, for example, joint staff appointments, exchange teaching assignments between staff members, the

* Symposium, Mathematical Association of America, September 2, 1952, Michigan State College.

familiarizing of staff members with the engineering curriculum on the one hand and the formal mathematics courses on the other, by the preparation and distribution of extensive course outlines and attendance in courses in other fields as visitors. All of these ways are being utilized but much could be gained by an increase in their use. Since mathematics takes up a large part of the generally crowded engineering curriculum (almost 25% of the first two years' work, measured in semester hours, at some schools), and since more and more engineers are taking more mathematics as undergraduates than in the past, and more and more engineers are prolonging their training to advanced degrees either at schools of engineering or in special programs set up by their employers, it is a fundamental problem of departments of mathematics to see to it that the most effective possible use is made of the time engineers devote to their mathematics courses.

The second speaker, Dr. C. H. Harry, Bureau of Aeronautics, Department of the Navy, spoke on *Mathematics used for engineering designs*. Dr. Harry maintained that the mathematical training of engineers should be based primarily on the fact that whenever an engineer at work employs mathematics he must ultimately arrive at a numerical answer. This is the case whether the numbers he seeks are, for example, merely the lumped constants of a simple electrical circuit which is to have some prescribed property, or the design parameters of a more complicated network or structure, or, at a still higher level, frequently referred to nowadays as *Operational Research*, numbers which represent the relative effectiveness of complicated systems involving machines or weapons, men, and, in military situations, tactics and strategy. It is immaterial to the engineer, and even more so to Management or Command, whether these numbers are arrived at ultimately by the substitution of numerical values of variables in "closed formulas," or otherwise, provided they are as accurate as the data permit, or as reliable as can be hoped for in the more complicated situations. In the simpler cases, the numbers will ultimately be checked by testing the circuit or network. In the more complicated cases of systems evaluation, the testing is much more difficult, as is the type of analysis involved. It is further characteristic of much of the mathematical work of an engineer that it is based largely on relatively elementary mathematical concepts, although more sophisticated mathematical ideas are finding more and more application at the present time.

Dr. Harry stated that in his opinion present mathematics courses for engineers do not give sufficient recognition to the numerical aspect of the engineers' mathematical work. Specifically, for instance, he maintained that in a first course in elementary differential equations, he would consider about two weeks as sufficient for the consideration of differential equations which can be solved in "closed form," and that the rest of the time should be devoted to solution by numerical methods. This point was seriously challenged by several of his listeners. While it was admitted that the engineer seeks always a numerical answer, it was claimed that the proper function of the mathematics department is to

acquaint the engineering student with basic mathematical concepts; that while, admittedly, most of the problems in our elementary mathematics texts used by engineers were not of the type the engineer would be called upon to solve at work, they were essential to the developing of his understanding of the mathematical concepts; that the mathematics departments would not have time in present courses to develop the mathematical concepts and introduce work on numerical methods as proposed by Dr. Harry. The controversy here was summarized as covering the relative emphasis which should be given in mathematics courses for engineers to the three aspects of mathematics itself, the linguistic, the logical, and the technical or computational, or, more briefly, between the conceptual and technical aspects. The resolution of the question should be a major concern of engineering faculties and teachers of mathematics. Probably some experimentation will be required before generally acceptable answers are arrived at. In this connection, a "two-track" system for engineers, one emphasizing the conceptual, the other the technical, aspects of mathematical training, is under consideration for an early trial in at least one school.

A DERIVATION OF THE FORMULAS FOR SIN $(\alpha+\beta)$ AND COS $(\alpha+\beta)$

A. K. BETTINGER, The Creighton University

The following procedure for deriving the addition formulas of plane trigonometry has been used for some time by the author. To the best of our knowledge it has not previously appeared in print.

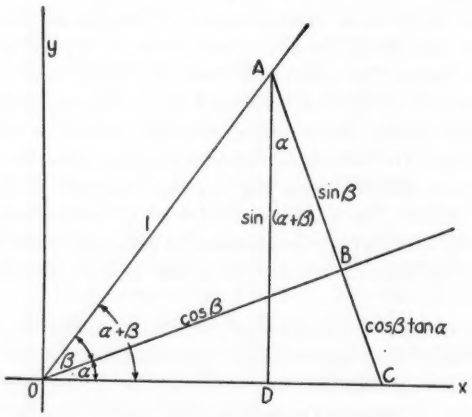


FIG. 1

We shall derive the formulas for the case when both α and β are acute angles and $\alpha+\beta<90°$.

Figure 1 will be used to prove the formula for sin $(\alpha+\beta)$.

In Figure 1, let $OA=1$, draw AC perpendicular to OB, and AD perpendicular to OX. Write the lengths of the sides on the figure as shown.

Now by definition the cosine of angle DAC equals DA/AC, that is,

$$\cos\alpha = \frac{\sin(\alpha+\beta)}{\cos\beta\tan\alpha+\sin\beta}.$$

Solving for $\sin(\alpha+\beta)$, we have

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\alpha(\cos\beta\tan\alpha+\sin\beta)\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta.\end{aligned}$$

Figure 2 is found more convenient for deriving the formula for $\cos(\alpha+\beta)$ than the preceding one. Hence we let $OA=1$, draw CA perpendicular to OB, and AD perpendicular to OY.

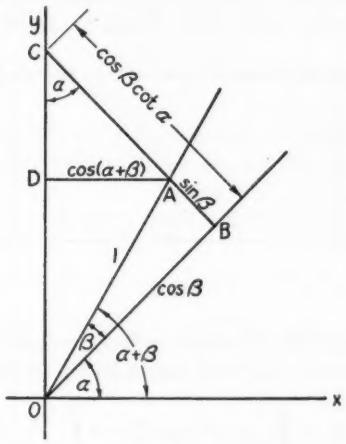


FIG. 2

By definition the cosine of angle DAC equals $\sin\alpha=DA/AC$, hence

$$\sin\alpha = \frac{\cos(\alpha+\beta)}{\cos\beta\cot\alpha-\sin\beta}.$$

Solving for $\cos(\alpha+\beta)$, we have

$$\begin{aligned}\cos(\alpha+\beta) &= \sin\alpha(\cos\beta\cot\alpha-\sin\beta)\\ &= \cos\alpha\cos\beta-\sin\alpha\sin\beta.\end{aligned}$$

The construction is readily extended to other quadrants for demonstrations in different cases. There is no need, however, for showing the construction in the numerous cases that may occur since the generality of the formulas does not

depend on the particular construction used to derive the formulas.

The validity of the formulas in general may be shown by a method similar to that in Todhunter's *Plane Trigonometry*, page 53.

A NOTE ON CAUCHY'S THEOREM

HARRY LASS, University of Illinois

The usual test for analyticity of a complex function $f(z)=u(x, y)+iv(x, y)$ consists in showing that the Cauchy-Riemann equations hold and that the first partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous. It seems desirable, therefore, to obtain a proof of one form of Cauchy's theorem by a direct method involving the above hypotheses on $f(z)$. We proceed to do this using a method due to Bliss* which avoids the use of Green's theorem.

Let S be a simply-connected open region such that the Cauchy-Riemann equations hold and such that the partial derivatives of u and v are continuous at every point of S. We first show that $\int f(z)dz$ vanishes for any rectangle in S.

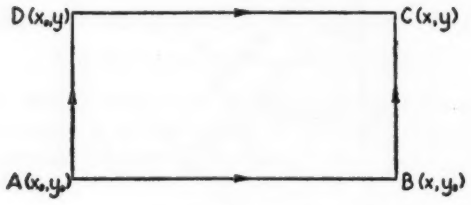


FIG. 1

Consider the rectangle $ABCD$ contained entirely in S, see Fig. 1. We can obtain a complex function of the two real variables x and y by defining

$$F(x, y) = \int_{x_0}^{x} f(t + iy_0)dt + i\int_{y_0}^{y} f(x + it)dt. \tag{1}$$

$F(x, y)$ is simply the sum of the integrals of $f(z)$ along the straight lines AB and BC. To obtain $\partial F/\partial x$ we note that

$$F(x + \Delta x, y) = \int_{x_0}^{x+\Delta x} f(t + iy_0)dt + i\int_{y_0}^{y} f(x + \Delta x + it)dt,$$

and that $\partial F/\partial x = \lim_{\Delta x \to 0}[F(x+\Delta x, y) - F(x, y)]/\Delta x$. We obtain

$$\frac{\partial F}{\partial x} = f(x + iy_0) + i\int_{y_0}^{y} \frac{\partial f(x + it)}{\partial x}\, dt \tag{2}$$

since $f(z)$ is continuous and $\partial f/\partial x$ is uniformly continuous on the closed straight

* American Mathematical Society Colloquium Publications, vol. XVI.

line segment BC. Similarly, $\partial F/\partial y = if(x+iy) = if(z)$.

We now define

$$G(x, y) = i\int_{y_0}^{y} f(x_0 + it)dt + \int_{x_0}^{x} f(t + iy)dt. \tag{3}$$

Note that $G(x, y)$ is the sum of the integrals of $f(z)$ along the straight line paths AD and DC. We easily obtain

$$\begin{aligned} \frac{\partial G}{\partial x} &= f(x + iy) = f(z) \\ \frac{\partial G}{\partial y} &= if(x_0 + iy) + \int_{x_0}^{x} \frac{\partial f(t + iy)}{\partial y}\, dt. \end{aligned} \tag{4}$$

To show that $\partial F/\partial x = \partial G/\partial x$ we observe that $\partial f/\partial x = (1/i)(\partial f/\partial y)$ from the Cauchy-Riemann equations. Thus (2) becomes

$$\frac{\partial F}{\partial x} = f(x + iy_0) + \int_{y_0}^{y} \frac{\partial f(x + it)}{\partial t}\, dt = f(x + iy) = f(z).$$

Similarly $\partial F/\partial y = \partial G/\partial y$. Thus $F = G + constant$ and since $F = G = 0$ at $A(x_0, y_0)$ we have $F \equiv G$. Hence $\int f(z)dz$ vanishes around the closed rectangular path $ABCD$.

We now consider any closed rectifiable Jordan curve, Γ, inside the rectangle $ABCD$, see Fig. 2.

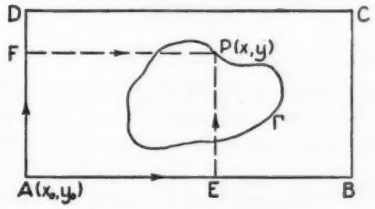


FIG. 2

Since $F \equiv G$ we have (considering the rectangle $AEPF$)

$$\begin{aligned} F(x, y) &= \int_{x_0}^{x} f(t + iy_0)dt + i\int_{y_0}^{y} f(x + it)dt \\ &= i\int_{y_0}^{y} f(x_0 + it)dt + \int_{x_0}^{x} f(t + iy)dt \end{aligned}$$

and $\partial F/\partial x = f(z)$, $\partial F/\partial y = if(z)$. Thus $dF = f(z)(dx + idy) = f(z)dz$ so that $\oint_\Gamma f(z)dz = \oint_\Gamma dF = \oint_\Gamma dU + i\oint_\Gamma dV = 0$, where $F(x, y) = U(x, y) + iV(x, y)$. The Riemann-Stieltjes integrals $\oint_\Gamma dU$, $\oint_\Gamma dV$ certainly vanish.

The final step proceeds as follows. Let C be any closed rectifiable Jordan curve in S. We apply a fine enough rectangular mesh on S so that the rectangles containing C lie inside S. This can be done since C is closed and S is open. The integration of $f(z)$ over all rectangles interior to C plus the integration of $f(z)$ over those boundaries which include C vanish from the preceding results. The sum of these integrations yields $\oint_C f(z)dz=0$.

AB AND *BA* HAVE THE SAME CHARACTERISTIC ROOTS*

JOEL BRENNER, State College of Washington

1. Introduction. The theory of matrices, and in particular the theory of characteristic roots of matrices, has now progressed far enough so that it should stand on its own feet, and be independent of the theory of determinants. One advantage of establishing an independent theory is this, that many theorems on matrices hold just as well whether the elements are numbers from a non-commutative or from a commutative field or even ring; but the theory of determinants over a non-commutative ring is not effective enough to furnish by itself proofs of those theorems.

A more practical advantage of a proof of a theorem on matrices independent of the theory of determinants is that it is simpler. The proof that the characteristic roots of the matrices AB and BA are the same is an illustration of this fact. One popular proof depends on the fact that the characteristic roots of AB are the solutions of the equation $\det(AB-\lambda I)=0$; if B is non-singular, the matrices $B(AB-\lambda I)B^{-1}$ and $(AB-\lambda I)$ have the same determinant, so the theorem follows. But if B is singular, this proof breaks down, and much more elaborate arguments are needed.

A shorter proof of this theorem is given below. To make this proof independent of any previous knowledge, it would be necessary first to prove that every matrix has a characteristic root, and second that every matrix B has a conjugate $T^{-1}BT$ in triangular form, with the characteristic roots ranged in any convenient order down the main diagonal. The first theorem can be proved simply and without the need for theorems from determinants [1]; the second theorem is very easy to prove; even with the additional restriction that the transforming matrix should be unitary the proof is not very long [2, p. 25].

2. The proofs. The following exposition makes no attempt at generality, but emphasizes method of argument.

Let A, B be $n\times n$ matrices over a field. By a characteristic root of B, we mean a number λ such that a non-null vector x exists with $Bx=x\lambda$.

LEMMA. *If T is non-singular, B and $T^{-1}BT$ have the same characteristic roots.*

* The writing of this article was sponsored by the Office of Ordnance Research.

Proof. If the relation $Bx=x\lambda$ holds, then the relation $T^{-1}BTy=y\lambda$ holds, and conversely. Here y is $T^{-1}x$, x is Ty, and if either x or y is not the null vector, the other is not the null vector.

THEOREM. *The matrices AB and BA have the same characteristic roots.*

Proof. By the lemma, we may suppose that B is triangular, and that the numbers in the main diagonal of B are so arranged that any zeros occur at the end.

If the relation $ABx=x\lambda$ holds, then the relation $BAy=y\lambda$ also holds, where y is the vector Bx. If the vector Bx is not zero, then λ is automatically a characteristic root of BA. On the other hand, if Bx is the null vector, then the equation $ABx=x\lambda$ reads $0=x\lambda$, so that λ is zero. We have to prove that zero is also a characteristic root of BA. Only in this part of the proof do we need to know that B can be transformed into triangular form. It is clear that B is singular (has no inverse), for otherwise the equation $Bx=0$ would be impossible with $x\neq 0$. Since B is singular, at least one characteristic root of B must be 0; for the inverse of a triangular matrix, no one of whose diagonal elements is 0, can easily be computed. Thus the last diagonal element of B is 0, so that the last row of B is zero. Hence the last row of BA is zero, and the last row of BAy is zero, whatever y. The matrix equation $BAy=y0$ amounts to a set of $n-1$ equations in n unknowns, so there is always a non-trivial solution [3, p. 3].

In the case of matrices over a non-commutative field, the results above hold, although the proof breaks down when $\lambda=0$. The proof does hold for quaternions, however.

References

1. J. L. Brenner, Matrices of Quaternions, Pacific Journal of Mathematics, vol. 1, 1951, pp. 329–335.
2. F. D. Murnaghan, The Theory of Group Representations, John Hopkins Press, 1938.
3. E. Artin, Galois Theory, Notre Dame, 1948.

PROOF OF THE FIRST MEAN VALUE THEOREM OF THE INTEGRAL CALCULUS

T. PUTNEY, State College of Washington

Professor A. A. Bennett in Volume 31 (1924), page 40, of this MONTHLY called attention to the didactic interest which attaches to a particular form of proof of the first mean value theorem of the differential calculus. A proof of the same character is to be found in the well-known book by Theodore Chaundy *The Differential Calculus* (Oxford University Press, 1935), p. 84. A precisely analogous argument can be employed to obtain a proof of the first mean value theorem of the integral calculus, and may be of interest in classroom instruction.

Define two functions $G(x)$ and $H(x)$ in $a\leqq x\leqq b$ by

$$G(x) \equiv \int_a^x f(t)g(t)dt, \qquad H(x) \equiv \int_a^x g(t)dt$$

where $f(x)$ and $g(x)$ are continuous in $a \leqq x \leqq b$ and $g(x)$ does not vanish in $a < x < b$. In the determinant

$$D(x) \equiv \begin{vmatrix} G(x) & H(x) & 1 \\ G(a) & H(a) & 1 \\ G(b) & H(b) & 1 \end{vmatrix} = \begin{vmatrix} G(x) & H(x) & 1 \\ 0 & 0 & 1 \\ G(b) & H(b) & 1 \end{vmatrix},$$

since $G(a) = H(a) = 0$. Furthermore, $D(a) = D(b) = 0$. Hence, by Rolle's theorem, $D'(\xi) = 0$ for some ξ, $a < \xi < b$ or

$$g(\xi) \begin{vmatrix} f(\xi) & 1 & 0 \\ 0 & 0 & 1 \\ G(b) & H(b) & 1 \end{vmatrix} = 0.$$

Finally, since $g(\xi) \neq 0$, this yields the desired result, namely $G(b) = f(\xi)H(b)$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1051. *Proposed by S. W. Golomb, Harvard University*

Given N objects and B boxes, what is a necessary and sufficient condition for at least two boxes to contain the same number of objects?

E 1052. *Proposed by H. H. Berry, U. S. Army Engineering Corps*

Let AOB be a fixed diameter of a given circle (O), and let P be any point on the circle. Denote by Q the foot of the perpendicular from P on AB and by R the foot of the perpendicular from O on PA. Let PQ and RO intersect in N, and let QR and PO intersect NA in L and M, respectively. Find the loci of points L, M, and N as P moves along the given circle.

E 1053. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Evaluate the determinant

$$\begin{vmatrix} x & x-1 & \cdots & x-k \\ x-1 & x & \cdots & x-k+1 \\ \cdot & \cdot & \cdots & \cdot \\ x-k & x-k+1 & \cdots & x \end{vmatrix}.$$

E 1054. *Proposed by F. D. Parker, St. Lawrence University*

Find the probability that the roots of the quadratic equation

$$x^2 + bx + c = 0, \qquad -k < b,\ c < k,$$

are imaginary.

E 1055. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The centers of the four circles passing through triples of vertices of a quadrangle $ABCD$ inscribed in (circumscribed about) a circle (O) are the vertices of a quadrangle $A'B'C'D'$ inscribed in (circumscribed about) a circle (O').

SOLUTIONS

An Invariant Determinant

E 1016 [1952, 328]. *Proposed by Norman Anning, University of Michigan*

Find the element of likeness in: (a) simplifying a fraction, (b) powdering the nose, (c) building new steps on the church, (d) keeping emeritus professors on campus, (e) putting B, C, D in the determinant

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ B & a^3 & 1 & a \\ C & D & a^3 & 1 \end{vmatrix}.$$

Solution by C. W. Trigg, Los Angeles City College. The value, $(1-a^4)^3$, of the determinant is independent of the values of B, C, and D. Hence, operation (e) does not change the value of the determinant but merely changes its appearance. Thus the element of likeness in (a), (b), (c), (d), and (e) is only that the appearance of the principal entity is changed. The same element appears also in: (f) changing the name-label of a rose, (g) writing a decimal integer in the scale of 12, (h) gilding the lily, (i) whitewashing a politician, and (j) granting an honorary degree.

Also solved by J. P. Ballantine, Julian Braun, I. A. Dodes, R. P. Eisinger, David Kerr and L. A. Ringenberg (jointly), David Loev, Charles Salkind, Alan Wayne, and the proposer.

A Property of Two Intersecting Circles

E 1017 [1952, 329]. *Proposed by Harry Furstenberg, New York City*

On the common secant AB of two intersecting circles, O and O', are chosen any two points, C and D, outside of either circle. The tangents CQ and CS are drawn to O' and O, respectively, on one side of AB and the tangents DR and DT are drawn to O' and O on the other side of AB. Prove that QR and ST intersect on AB.

Solution by A. Sisk, Maryville, Tenn. (The references in the following solution are to Altshiller-Court, *College Geometry*, 2nd ed., p. 198.)

The common chord AB is the radical axis of circles O' and O. Points Q and S are antihomologous points on O' and O respectively, as also are points R and T (435). Therefore, by definition, RQ and ST are antihomologous chords of O' and O. It follows that RQ meets ST on AB (434).

Also solved by I. A. Dodes, A. L. Epstein, E. I. Gale, L. M. Kelly, W. H. Mead, Jr., Margaret Olmsted, and the proposer.

A Minimum Value

E 1018 [1952, 329]. *Proposed by B. B. Misra, Ravenshaw College, Cuttack, India*

AB is a line segment, which is divided by the point O in the ratio $m:n$. P is a variable point (r, θ) referred to O as pole and OB as initial line. Subject to the conditions

$$0 \leqq r \leqq a < \min(AO, OB), \qquad 0 \leqq \theta \leqq \pi,$$

find the minimum value of $m^2/AP+n^2/BP$, and also find the position of P for which this minimum value is attained.

Solution by I. A. Dodes, Stuyvesant High School, New York City. Let us designate AP by p and BP by q and set

$$z = m^2/p + n^2/q.$$

Since, for fixed θ, p and q both increase when r increases, it follows that $r=a$ at $z_{\min}$ and that $z=z(\theta)$. Hence, for a relative minimum, we must have

$$dz/d\theta = -(m/p)^2 dp/d\theta - (n/q)^2 dq/d\theta = 0.$$

But, setting $AO=km$ and $OB=kn$, we have

$$p^2 = a^2 + k^2m^2 + 2akm\cos\theta,$$
$$q^2 = a^2 + k^2n^2 - 2akn\cos\theta,$$

whence

$$dp/d\theta = -(akm\sin\theta)/p, \qquad dq/d\theta = (akn\sin\theta)/q.$$

Substituting in $dz/d\theta = 0$ we find

$$p/q = m/n.$$

It follows that the sought point P is an intersection of the circle of radius a and center O with the circle of Apollonius for the segment AB and the ratio $m:n$. Since PO bisects angle APB we have

$$pq = a^2 + k^2mn.$$

It now follows that, for minimum z,

$$p^2 = m(a^2 + k^2mn)/n, \qquad q^2 = n(a^2 + k^2mn)/m,$$

whence an easy calculation gives

$$z_{\min} = \sqrt{mn}\,(m + n)/\sqrt{a^2 + k^2mn}.$$

Also solved by Julian Braun, L. A. Ringenberg, O. E. Stanaitis, and J. A. Tierney.

A Uniformly Convergent Sequence of Continuous Functions

E 1019 [1952, 329]. *Proposed by A. E. Livingston and L. H. Wegner, University of Oregon*

Let $\{f_n(x)\}$ be a sequence of functions continuous on the closed interval $[a, b]$, and let $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on the open interval (a, b). Then there exists a function $F(x)$ satisfying: (1) $F(x)$ is continuous on $[a, b]$, (2) $F(x) = f(x)$ for $x\epsilon(a, b)$, (3) $\lim_{n\to\infty} f_n(a) = F(a)$ and $\lim_{n\to\infty} f_n(b) = F(b)$.

Solution by L. A. Ringenberg, Eastern Illinois State College. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on (a, b), there is an integer $N > 0$ such that

$$n \geqq N \text{ and } x\epsilon(a, b) \text{ imply } |f_n(x) - f(x)| < \epsilon. \tag{1}$$

Let $x_0\epsilon(a, b)$ be given. Since $f_N(x)$ is continuous on $[a, b]$, there is a real number $\delta > 0$ such that

$$x\epsilon(a, b) \text{ and } |x - x_0| < \delta \text{ imply } |f_N(x) - f_N(x_0)| < \epsilon. \tag{2}$$

Let x be a real number such that $x\epsilon(a, b)$ and $|x - x_0| < \delta$. Then it follows from (1) and (2) that

$$\begin{aligned} |f(x) - f(x_0)| &\leqq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| \\ &\quad + |f_N(x_0) - f(x_0)| < 3\epsilon. \end{aligned} \tag{3}$$

Let $F(x) = f(x)$ for every $x\epsilon(a, b)$. In view of (3), $F(x)$ is continuous on (a, b).

For $i = 1, 2$, let n_i be any integer such that $n_i \geqq N$. Since $f_{n_i}(x)$ is continuous on $[a, b]$, there is a real number $\Delta > 0$ such that

$$0 < x - a < \Delta \text{ implies } |f_{n_i}(x) - f_{n_i}(a)| < \epsilon \text{ for } i = 1, 2. \tag{4}$$

Let x be a real number such that $0<x-a<\Delta$. Then, applying (1) and (4), we have

$$\begin{aligned}|f_{n_1}(a)-f_{n_2}(a)| &\leqq |f_{n_1}(a)-f_{n_1}(x)|+|f_{n_1}(x)-f(x)| \\ &\quad +|f(x)-f_{n_2}(x)|+|f_{n_2}(x)-f_{n_2}(a)|<4\epsilon.\end{aligned}$$

Thus the sequence $\{f_n(a)\}$ satisfies Cauchy's internal criterion for convergence. Let $F(a)=\lim_{n\to\infty} f_n(a)$. Let $M\geqq N$ be any integer such that

$$|f_M(a)-F(a)|<\epsilon. \tag{5}$$

Since $f_M(x)$ is continuous at $x=a$, there is a positive real number ρ such that

$$0<x-a<\rho \text{ implies } |f_M(x)-f_M(a)|<\epsilon. \tag{6}$$

Let x be a real number such that $0<x-a<\rho$. Then it follows from (5) (6), and (1) that

$$\begin{aligned}|F(a)-F(x)| &= |F(a)-f(x)| \\ &\leqq |F(a)-f_M(a)|+|f_M(a)-f_M(x)|+|f_M(x)-f(x)|<3\epsilon.\end{aligned}$$

Therefore $F(x)$ is continuous at $x=a$. Similarly $F(x)$ is continuous at $x=b$ if we define $F(b)=\lim_{n\to\infty} f_n(b)$.

Also solved by Harold Johnson, Azriel Rosenfeld, W. R. Smythe, Jr., O. E. Stanaitis, and the proposers.

Johnson pointed out that the desired results are found on pp. 152 and 154 of Carslaw, *Theory of Fourier's Series and Integrals*.

Traveling by the "Ride and Tie" Plan

E 1021 [1952, 407]. *Proposed by W. R. Ransom, Tufts College*

In the "ride and tie" plan (where the first man rides a horse for a stated time and leaves it for a second man, walking, to ride for a like time while the first man walks on) show that the rate of progress is the harmonic mean between the rates of riding and walking, and find for what fraction of the time the horse can rest.

Solution by Julian Braun, Washington, D. C. Let r and w be the riding and walking rates, respectively. At the beginning and end of a "cycle" of time duration t, the two men are together and the horse has rested for a length of time τ. The distance traversed by walking equals the distance traversed by riding, that is, $w(t+\tau)=r(t-\tau)$, whence the fraction of time the horse can rest is

$$\tau/t=(r-w)/(r+w).$$

The rate of progress is thus given by

$$w(t+\tau)/t=w\{1+(r-w)/(r+w)\}=2rw/(r+w).$$

Also solved by A. G. Anderson, Leon Bankoff, I. A. Dodes, A. L. Epstein,

Vern Hoggatt, Sr. and Vern Hoggatt, Jr. (jointly), M. S. Klamkin, J. D. E. Konhauser, L. J. Lander, David Mandelbaum, Margaret Olmsted, Azriel Rosenfeld, C. A. Swanson, V. V. Rao, and the proposer.

Klamkin remarked that for m men and one horse the rate of progress is given by

$$mrw/[(m-1)r+w],$$

and the fraction of time the horse can rest is given by

$$[(m-1)r-w]/[(m-1)r+w].$$

The case of m men and h horses, each with a different speed, is a much more involved problem.

A Class of Heronian Triangles

E 1022 [1952, 407]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find all triangles for which the three sides and one of the altitudes are, in some order, measured by four integers in arithmetic progression.

Solution by R. W. Huff, College of Wooster. Representing the sides in order of decreasing length and the altitudes to these sides by a, b, c, h_a, h_b, h_c, respectively, and noting that the altitude to any side is smaller than or equal to (the latter prohibited by the conditions of the problem) either of the remaining sides, we arrive at the following four cases for the problem: (I) a, b, c, h_a, (II) a, b, c, h_b, (III) a, b, c, h_c, (IV) a, b, h_c, c, where the segments are given in order of decreasing length. We now express the four segments concerned in terms of a and the common difference d, and substitute the results in the equations

$$(ih_i)^2 = 4s(s-a)(s-b)(s-c), \qquad i = a, b, c,$$

where $2s=a+b+c$. In solving, the only valid solution is found in case (II), where $a=15d$. Hence the only triangles satisfying the problem are: $a=15d$, $b=14d$, $c=13d$, $h_b=12d$, $d=1, 2, 3, \cdots$.

Also solved by Leon Bankoff, N. G. Gunderson, Sam Kravitz, F. D. Parker, Azriel Rosenfeld, E. P. Starke, C. W. Trigg, and the proposer.

The interested reader might like to consult E 695 [1946, 334].

Application of the Maximum-Modulus Theorem

E 1023 [1952, 407]. *Proposed by D. J. Newman, Harvard University, and H. S. Shapiro, Massachusetts Institute of Technology*

(1) Given any set of points in the plane, not all coincident with the origin, show that there exists a point on the unit circle such that the product of the distances from it to these points is greater than 1.

(2) Given any set of points on the unit circle, not the vertices of a regular polygon, show that there exists a point on the unit circle such that the product of the distances from it to these points is greater than 2.

Solution by Fritz Herzog, Michigan State College. (1) Let the given points be denoted by $z_1, z_2, \cdots, z_n$. Put

$$P(z) \equiv (z - z_1)(z - z_2) \cdots (z - z_n)$$
$$= z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n.$$

We are to show that, for all values of z with $|z| = 1$, $\max |P(z)| \geqq 1$, equality holding if and only if $z_1 = z_2 = \cdots = z_n = 0$, that is, if and only if $P(z) \equiv z^n$. (In the following the maximum is always taken over all values of z with $|z| = 1$.)

Let

$$Q(z) \equiv z^n P(1/z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + 1.$$

Obviously, $\max |Q(z)| = \max |P(z)|$. But $Q(0) = 1$ and hence, by the maximum-modulus theorem, $\max |Q(z)| \geqq 1$, equality holding if and only if $Q(z) \equiv 1$, that is, if and only if $P(z) \equiv z^n$.

(2) We use the above notation and assume that

$$|z_1| = |z_2| = \cdots = |z_n| = 1.$$

As above, we are to show that $\max |P(z)| = \max |Q(z)| \geqq 2$, equality holding if and only if the z_k are the vertices of a regular n-gon, that is, if and only if $c_1 = c_2 = \cdots = c_{n-1} = 0$.

Assume first that $\max |Q(z)| < 2$. Then $|Q(z)| < 2$ for $|z| \leqq 1$, and hence all zeros of

$$Q(z) - 2 = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z - 1$$

lie in $|z| > 1$, which is incompatible with the fact that

$$|c_n| = 1. \tag{I}$$

Assume secondly that $\max |Q(z)| = 2$. Then $|Q(z)| < 2$ for $|z| < 1$, and hence all zeros of $Q(z) - 2$ must lie in $|z| \geqq 1$ and therefore, by (I), on $|z| = 1$. Let these zeros be denoted by $u_1, u_2, \cdots, u_n$, and let $v_1, v_2, \cdots, v_n$ be the zeros of $Q(z)$ itself; note that the v_m are the reciprocals of the z_m and hence also lie on $|z| = 1$. It is obvious that $u_k \neq v_m$ for all k and m. Let U be the polygon that has the u_k as vertices and V the same for the v_m. Then* the zeros of $Q'(z)$ must lie in the intersection of U and V and hence in $|z| < 1$. It follows that none of the quantities $Q'(u_k)$ and $Q'(v_m)$ vanishes, so that $u_i \neq u_j$ and $v_i \neq v_j$ for $i \neq j$. Let

$$R(z) \equiv c_n z^{n-1} + c_{n-1} z^{n-2} + \cdots + c_2 z + c_1,$$

so that $zR(z) = Q(z) - 1$. We then have $u_k R(u_k) = 1$ and $v_m R(v_m) = -1$, hence $|R(z)| = 1$ at $z = u_k$ and $z = v_m$. Now $|R(e^{i\theta})|^2$ is a trigonometric polynomial in θ, with real coefficients, of degree no higher than $n - 1$, and assuming the value

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* See G. Polya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Section III, Problem 31.

1 at $2n$ distinct points of the interval $0 \leqq \theta \leqq 2\pi$. We conclude† that $|R(e^{i\theta})|^2 \equiv 1$. Considering the constant term of $|R(e^{i\theta})|^2$, we thus obtain

$$|c_n|^2 + |c_{n-1}|^2 + \cdots + |c_2|^2 + |c_1|^2 = 1,$$

and hence, by (I), $c_{n-1} = c_{n-2} = \cdots = c_2 = c_1 = 0$.

Also solved by the proposers. Shapiro further proposed the following similar problem:

(3) Let $P_1, \cdots, P_n$ denote n points on a circle of unit radius, and set τ_k equal to the product of the distances from P_k to the other points. Then

$$\sum_{k=1}^{n} (1/\tau_k) \geqq 1,$$

equality being attained if and only if the n points form a regular n-gon.

Editorial Note. One wonders if purely geometric solutions can be supplied for these problems. The solutions by function theory do not extend themselves even to three dimensions. Problem (1) is not true for one dimension; is it true for any dimension higher than two? Can problems (2) and (3) be extended, in part at least, to higher dimensions?

Concerning the Four Color Theorem

E 1024 [1952, 407]. *Proposed by David Gale, Brown University*

Assuming that the four color theorem is true for planar maps containing a finite number of regions, show that it is also true for planar maps containing an infinite number of regions.

Solution by the Proposer. Let M be the given infinite map. The proof is simplified by using the fact that such a map contains only a countable number of regions. Let the regions be denoted by $R_1, R_2, \cdots$, and let $M_1, M_2, \cdots$ denote the finite sub-maps of M defined by $M_n = R_1 \cup R_2 \cup \cdots \cup R_n$. For each M_n let C_n denote a particular admissible coloring of M (which by assumption exists). Finally, for $i \leqq n$ let $c_{n,i}$ denote the color assigned to the region R_i by the coloring C_n.

We now describe inductively a procedure for coloring M. Let Γ denote the collection of all the colorings, C_n. Consider now the colors $c_{n,i}$. Since there are only a finite number (four) of colors, it follows that one can choose a color c_1 such that $c_1 = c_{n,1}$ for infinitely many n. Now let Γ_1 denote the set of all colorings of Γ for which $c_{n,1} = c_1$. Note that the collection Γ_1 is infinite, so that one can repeat the process, that is, choose a color c_2 such that $c_{n,2} = c_2$ for infinitely many $C_n \epsilon \Gamma_1$, and let Γ_2 be the set of all $C_n \epsilon \Gamma_1$ such that $c_{n,2} = c_2$. By repetition in this way we obtain an infinite sequence of colors, $c_1, c_2, \cdots$, and corresponding infinite subcollections $\Gamma_1, \Gamma_2, \cdots$, of Γ. We now define a coloring C_∞ of M by assigning to R_i the color c_i. To show that C_∞ is admissible, let R_j and R_k be two

† See G. Polya and G. Szegö, *loc. cit.*, vol. 2, Section VI, Problem 14.

contiguous regions of M with $j<k$. By examining the coloring process above it is seen that c_j and c_k are exactly the colors assigned to R_j and R_k by all of the colorings of Γ_k. But since each of these colorings is admissible it follows that $c_k \neq c_j$, which proves the theorem.

The above proof will be recognized as essentially an application of the Cantor diagonal process.

Also solved by S. W. Golomb, who called attention to the article by C. N. Reynolds, *The present status of the four color problem*, Bull. Am. Math. Soc., March, 1952, p. 207, abstract 263.

Harold Kuhn remarked that the problem was stated and solved by D. König, *Über eine Schlussweise aus dem Endlichen ins Undendliche*, Acta Szeged 3 (1927), §3.

Some Finite Series Inequalities

E 1025 [1952, 407]. *Proposed by K. L. Chung, Cornell University*

Show that if

$$a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$$

and

$$\sum_{j=1}^{k} a_j \leqq \sum_{j=1}^{k} b_j$$

for $k=1, \cdots, n$, then

$$\sum_{j=1}^{n} a_j^2 \leqq \sum_{j=1}^{n} b_j^2,$$

equality holding if and only if $a_j=b_j$, $j=1, \cdots, n$.

Solution by G. A. Hunt, Cornell University. We have

$$(a_k - a_{k+1}) \sum_{j=1}^{k} a_j \leqq (a_k - a_{k+1}) \sum_{j=1}^{k} b_j$$

for $k=1, \cdots, n$ and with $a_{n+1}=0$. Summing each side from $k=1$ to $k=n$ and using Abel's partial summation formula we obtain

$$\sum_{j=1}^{n} a_j^2 \leqq \sum_{j=1}^{n} a_j b_j.$$

An application of Schwarz's inequality now gives the result, which can obviously be extended to the case of infinite series.

Also solved by Stephen Chase, W. H. J. Fuchs, Arthur Gregory, Cecil Hastings, Jr., A. E. Livingston, M. Perisastri, J. B. Rosser, O. E. Stanaitis, and R. J. Walker.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4523. *Proposed by H. S. M. Coxeter, University of Toronto*

Given a point P in real projective space, and four skew lines a, b, c, d, find a linear construction for the line through P of the linear congruence determined by a, b, c, d.

4524. *Proposed by E. C. Milner, University of Malaya, Singapore*

Prove that for any positive integers n, N there are blocks of consecutive integers of length greater than N, with the property that each of their totients is divisible by n.

4525. *Proposed by Paul Erdös, National Bureau of Standards, Los Angeles*

Let $u_1<u_2<\cdots$ be the sequence of integers all of whose prime factors have exponents exceeding one. Prove that the density of integers of the form u_i+u_j is zero (*i.e.* the number of integers $\leqq x$ of the given form is $<\epsilon x$ for any ϵ if x is large enough).

4526. *Proposed by Peter Ungar, New York University*

Show that for any $k>1$ there exist k-chromatic graphs which contain no three mutually connected nodes.

Note. A linear graph is the figure obtained by joining certain parts of n points by line segments. If it requires k colors to so color the segments that no two ends of the same color meet at a node, the graph is k-chromatic.

4527. *Proposed by D. J. Newman, Harvard University*

Is there a function $f(x)$ continuous in the closed interval (0, 1) and such that in this interval $f(x)+f(x^2)=x$?

SOLUTIONS

Convergence of an Erdös Sequence

4460 [1951, 636]. *Proposed by Paul Erdös, National Bureau of Standards, Los Angeles, Cal.*

Let $f(1)=1$, $f(n)=(1+c_1)f(n-1)-c_2f(n-2)-\cdots-c_{n-1}f(1)$, $c_i\geqq 0$, $i=1, 2, \cdots$, $c_1=c_2+c_3+\cdots$. What is the necessary and sufficient condition for the convergence of $f(n)$?

Solution by Michael Golomb, Purdue University. Let $P(z)$ denote the function represented by the power series.

$$1 - (1 + c_1)z + c_2z^2 + c_3z^3 + \cdots = P(z).$$

Since $0 \leqq c_i \leqq c_1$, the series converges for $|z| \leqq 1$, and since $c_1 = c_2 + c_3 + \cdots$ we have $P(z) = (1-z)Q(z)$ where

$$Q(z) = 1 - c_2z - c_3z(1 + z) - c_4z(1 + z + z^2) - \cdots .$$

For sufficiently small z, $1/P(z)$ can be expanded as a power series, $\sum a_kz^k$, where $a_0 = 1$ and

$$a_k - (1 + c_1)a_{k-1} + c_2a_{k-2} + \cdots + c_ka_0 = 0.$$

From the definition of $f(n)$ it is seen that $f(n) = a_{n-1}$. Hence

$$\sum_{n=1}^{\infty} f(n)z^{n-1} = 1/P(z), \qquad \sum_{n=1}^{\infty} f(n)z^n = z/P(z),$$

$$f(1) + \sum_{n=1}^{\infty} \{f(n + 1) - f(n)\}z^n = \frac{1 - z}{P(z)} = \frac{1}{Q(z)}.$$

It follows that

$$\lim_{n\to\infty} f(n) = f(1) + \sum_{n=1}^{\infty} \{f(n + 1) - f(n)\}$$

exists if and only if the power series for $1/Q(z)$ converges for $z = 1$. Hence it is necessary that $Q(z) \neq 0$ for $z = x$ real, $0 \leqq x \leqq 1$, and this condition is also sufficient since $|Q(z)| \geqq 1 - c_2x - c_3x(1+x) - \cdots$ for $|z| \leqq x$. Since $Q(x)$ is monotone non-increasing and $Q(0) = 1$ we have $Q(x) \neq 0$ for $0 \leqq x \leqq 1$ if and only if $Q(1) > 0$, that is, if

$$C = c_2 + 2c_3 + 3c_4 + \cdots < 1.$$

This then is the necessary and sufficient condition for the convergence of $f(n)$. If this condition is satisfied $\lim f(n) = 1/Q(1) = 1/C$.

Also solved by R. P. Agnew, Robert Breusch, Einar Hille, D. J. Newman, H. N. Shapiro, and Gabor Szegö.

Probabilities for a Die of Irregular Shape

4461 [1951, 636]. *Proposed by George Grossman, De Witt Clinton High School, New York City*

A die is made of homogeneous material in the shape of a rectangular parallelepiped and is tossed onto a level surface from which it will not bounce. What is the probability that a particular face will come up on top?

Solution by H. C. Kranzer, Student, New York University. We treat the more general situation where the die may have the shape of any polyhedron and may be loaded in any manner, subject only to the restriction that it be capable of resting upon any of its faces. This last restriction is equivalent to the condition that the polyhedron be convex and that the foot of the perpendicular from the center of mass to the plane of any face lie within that face.

Suppose such a die is tossed onto a level surface in such a way that it will not bounce or roll. It is evident that if, at the moment of first contact between any part of the die and the surface, we drop a plumb line from the center of mass, the face through which this plumb line passes will be on bottom when the die has come to rest. Since the orientation of the die at the moment of first contact is perfectly at random, we may equally well choose a fixed orientation in space for the die and take an arbitrary random direction for the vertical. This shows at once the principal result: *The probability of any given face coming out on bottom is proportional to the solid angle subtended by that face at the center of mass.*

We may find the probability explicitly by the following method: Consider an elementary cone of volume dV, solid angle dw, having its vertex at C the center of mass, and having for its base a region of area dA of the face in question. Let r be the distance of this elementary region from C, and let h be the perpendicular distance from C to the plane of the face. Then $dV=r^3dw/3=hdA/3$; hence $dw=hdA/r^3$ and the required solid angle w is given by

$$w = h\iint dA/r^3,$$

where the integration is performed over the entire area of the face. The probability that the face come out on bottom is then given by

$$p = \frac{w}{4\pi} = \frac{h}{4\pi}\iint \frac{dA}{r^3}.$$

In the special case of a homogeneous rectangular parallelepiped, the probability of a face coming up on top is the same as that of its complementary face coming out on bottom. We will suppose that the required face has sides $2a$ and $2b$, and that the other dimension of the parallelepiped is $2c$. Then C is at the geometrical center, and $h=c$. If we take polar coördinates with the origin at the center of the face and the polar axis parallel to side b, we have $dA=\rho d\rho d\theta$ and $r=(c^2+\rho^2)^{\frac{1}{2}}$. Then

$$p = \frac{1}{\pi}\arcsin\frac{ab}{\sqrt{(a^2+c^2)(b^2+c^2)}}.$$

If $a=b=c$, this reduces to the expected value 1/6.

Also solved by F. G. Fender.

Editorial Note. J. G. Diaz proposes consideration of a die in the shape of a right prism whose bases are regular pentagons. J. H. Braun proposes a cuboctahedron. If the requirement of no bouncing or rolling is removed, the problem becomes much more difficult. Braun notes that in this case the answer cannot be based (solely) on the proportionate solid angle subtended by the faces at the centroid to the whole solid angle owing to the fact that some faces are nearer to the centroid than others, which reduces the tendency of the more remote faces to "show."

A Double Series Summation

4462 [1951, 636]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, New York*

If

$$S_{2m} = \frac{1}{2^{2m}} + \frac{1}{4^{2m}} + \frac{1}{6^{2m}} + \cdots,$$

find the value of

$$\frac{S_2}{2\cdot 3} + \frac{S_4}{4\cdot 5} + \frac{S_6}{6\cdot 7} + \cdots.$$

I. *Solution by V. C. Harris, San Diego State College, California.* If we let

$$T_{2m} = \frac{1}{2\cdot 3}\left(\frac{1}{2m}\right)^2 + \frac{1}{4\cdot 5}\left(\frac{1}{2m}\right)^4 + \frac{1}{6\cdot 7}\left(\frac{1}{2m}\right)^6 + \cdots,$$

then the value of the given expression is the same as that of $\sum_{m=1}^{\infty} T_{2m}$. By integrating $r/(1-r^2) = r + r^3 + r^5 + \cdots$, $|r| < 1$, from 0 to r twice and then setting $r = 1/2m$, we find

$$T_{2m} = 1 - \frac{2m+1}{2}\log\frac{2m+1}{2m} + \frac{2m-1}{2}\log\frac{2m-1}{2m},$$

$m = 1, 2, 3, \cdots$, whence

$$\sum_{m=1}^{n} T_{2m} = n + \log\frac{2^n(n!)}{(2n+1)^{n+1/2}} = \log\frac{2^n e^n (n!)}{(2n+1)^{n+1/2}}.$$

Using Stirling's formula, $\lim e^n n!/n^{n+\frac{1}{2}} = \sqrt{2\pi}$, we obtain

$$\sum_{m=1}^{\infty} T_{2m} = \tfrac{1}{2}\log(\pi/e).$$

II. *Solution by Leonard Carlitz, Duke University.* Starting with the familiar formula

$$\sin x = x\prod_{m=1}^{\infty}\left(1-\frac{x^2}{m^2\pi^2}\right),$$

we have

$$\log\sin x = \log x - \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\frac{x^{2k}}{m^{2k}\pi^{2k}},$$

$$\log\sin\frac{x}{2} = \log\frac{x}{2} - \sum_{k=1}^{\infty}\frac{1}{k}\frac{x^{2k}}{\pi^{2k}}S_{2k}.$$

Integration yields

$$\int_0^{\pi}\log\sin\frac{x}{2}\,dx = \pi\left(\log\frac{\pi}{2}-1\right) - \pi\sum_{k=1}^{\infty}\frac{S_{2k}}{k(2k+1)}.$$

Since the definite integral is known to have the value $-\pi\log 2$, the desired summation reduces to $\frac{1}{2}(\log\pi - 1)$.

Glaisher (*Messenger of Mathematics*, v. 44, 1914, 1–10) has proved a number of analogous results. In particular the following may be mentioned.

$$\frac{S_2}{3\cdot4}+\frac{S_4}{5\cdot6}+\frac{S_6}{7\cdot8}+\cdots=\frac{1}{4},$$

$$\frac{S_2}{4\cdot5\cdot6}+\frac{S_4}{6\cdot7\cdot8}+\frac{S_6}{8\cdot9\cdot10}+\cdots=\frac{1}{48},$$

$$\frac{S_3}{4\cdot5\cdot6}+\frac{2S_5}{6\cdot7\cdot8}+\frac{3S_7}{8\cdot9\cdot10}+\cdots=\frac{1}{24},$$

$$\frac{S_3}{3\cdot4}+\frac{S_5}{5\cdot6}+\frac{S_7}{7\cdot8}+\cdots=\frac{1}{2}-\frac{1}{2}\gamma,$$

where γ denotes Euler's constant. In connection with the last see also Ramanujan, *Messenger of Mathematics*, v. 46, 1917, 11–21 (or *Collected Papers*, 163–168).

Also solved by T. M. Apostol, Robert Breusch, Calvin Foreman, Samuel Goldberg, Emil Grosswald, Hyman Kaufman, M. S. Klamkin, M. Perisastri, Azriel Rosenfeld, P. C. Sikkema, Michael Skalskyj, O. E. Stanaitis, Chih-yi Wang, R. E. Wild, G. T. Williams, and the Proposer.

A Möbius Involution

4463 [1951, 702]. *Proposed by Roland Deaux, Faculté Polytechnique, Mons, Belgium*

The lines AP, BP, CP drawn through the vertices and a given point P in the plane of triangle ABC meet the opposite sides in A_1, B_1, C_1. Let A', B', C'

be the points which with A, B, C respectively separate harmonically the pairs (B_1, C_1), (C_1, A_1), (A_1, B_1). Prove that the pairs (A, A'), (B, B'), (C, C') belong to a Möbius involution.

Solution by the Proposer. Let F, F' be the foci of the conic section γ which touches in A_1, B_1, C_1 the sides BC, CA, AB. Siebeck's theorem* gives the direct circular transformation

$$\omega = \begin{pmatrix} F & F' & B & C_1 & A \\ F & F' & C & A & B_1 \end{pmatrix}$$

to which the Möbius involution

$$I_a = (F\ F',\ A\ A,\ B_1\ C_1)$$

is harmonic; A' is thus the second fixed point of I_a. Similarly we obtain the Möbius involutions

$$I_b = (F\ F',\ B\ B,\ B'\ B',\ C_1\ A_1), \qquad I_c = (F\ F',\ C\ C,\ C'\ C',\ A_1\ B_1),$$

whence the following Möbius involution

$$I = (F\ F,\ F'\ F',\ A\ A',\ B\ B',\ C\ C').$$

Note. Since γ and I have the same center O, the angles (OA, OA'), (OB, OB'), (OC, OC') have the same bisectors and

$$|OA \cdot OA'| = |OB \cdot OB'| = |OC \cdot OC'| = OF^2.$$

Suggested by the Curvature and Arc-length of a Circle

4464 [1951, 702]. *Proposed by R. Kissling, University of California, Berkeley*

Consider the class of single-valued differentiable functions $f(x)$ on the interval $0 \leqq x < 1$ such that $f(0) = 0$; $f(x) \to \infty$ as $x \to 1^-$. Let

$$K(x) = f'(x)/\{1 + [f(x)]^2\}^{3/2}, \qquad |K(x)| \leqq 1,\ 0 \leqq x < 1.$$

Prove that for any function of this class

$$\int_0^1 \{1 + [f(x)]^2\}^{1/2} dx = \pi/2.$$

I. *Solution by J. A. Tierney, United States Naval Academy.* We have

$$\int_0^1 K(x)dx = \lim_{\epsilon \to 0} \frac{f}{(1+f^2)^{1/2}}\Bigg]_{x=0}^{x=1-\epsilon} = \lim_{f \to \infty} \frac{f}{(1+f^2)^{1/2}} = 1.$$

It follows from $|K| \leqq 1$ on $[0, 1)$ that $K \equiv 1$ almost everywhere on $[0, 1)$. Hence

* See R. Deaux, *Introduction à la Géométrie des nombres complexes*, p. 125 (Brussels, 1947).

$$\int_0^1 (1+f^2)^{1/2}dx = \int_0^1 \frac{f'dx}{K(1+f^2)}$$

has the same value as

$$\int_0^1 \frac{f'dx}{1+f^2} = \tan^{-1} f\Big]_{x=0}^{x=1} = \pi/2.$$

II. *Solution by F. Bagemihl and W. Seidel, University of Rochester.* Let $g(x)=f(x)\{1+[f(x)]^2\}^{-1/2}$, $0\leqq x<1$, $g(1)=1$. Then $g(x)$ is continuous for $0\leqq x<1$; $g'(x)=K(x)$, and therefore $|g'(x)|\leqq 1$ for $0<x<1$; $g(0)=0$, $g(1)=1$. A simple application of the mean value theorem yields $g(x)=x$, and hence

$$f(x) = \pm\, x(1-x^2)^{-1/2}, \qquad 0 \leqq x < 1,$$

from which the relation to be proved follows immediately.

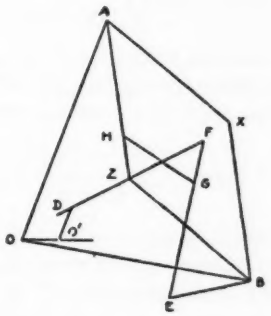
Also solved by J. W. Baldwin, A. B. Farnell, W. Fraser, Emil Grosswald, H. J. Hamilton, P. G. Kirmser, A. M. Peiser and Stanley Katz, J. H. Roberts, O. E. Stanaitis, and the Proposer.

A Conicograph

4465 [1951, 702]. *Proposed by E. I. Gale, University of Saskatchewan*

AXBZ is a jointed rhombus with sides of length $4a$. (See figure. For convenience a is taken considerably greater than half the unit.) *AO* and *BO* are bars of equal length. The fixed centers are *O* and *O'* with *OO'* half a unit. Let $O'D=\frac{1}{2}$, $FE=4a$, $FZ=HG=BE=2a$, $HZ=FG=a$. *D* is an adjustable set screw on bar *FZD* so that the length *ZD* can be set at pleasure.

Show that as *D* moves in a circle about *O'*, *X* describes the general conic of eccentricity $1/ZD$.



Solution by R. K. Morley, Worcester Polytechnic Institute. Figure *HZBEGF* constitutes a standard Kempe angle bisector. That is, quadrilaterals *HZFG* and *FZBE*, having sides proportional and common angle *ZFG*, and having cor-

responding parts similarly placed, are similar. Hence angles FZB and FZH are equal. Now the Peaucellier cell $OAXBZ$ is symmetrical about OX, so that FD produced passes through O and X.

Also in the Peaucellier cell $OAXBZ$

$$OZ \cdot OX = OA^2 - AX^2 = R^2 \text{ a constant.} \tag{1}$$

Let polar coördinates ρ and θ be OX and angle $O'OX$ respectively. Since $OO' = O'D = \frac{1}{2}$, we have $OD = \cos\theta$ and

$$OZ = DZ + OD = DZ + \cos\theta.$$

By (1)

$$\rho = OX = \frac{R^2}{OZ} = \frac{R^2}{DZ + \cos\theta} = \frac{R^2/DZ}{1 + (1/DZ)\cos\theta}.$$

which is the equation of a conic with eccentricity $1/DZ$. (Note the relation between the different types of conic and the types of limacon traced by point Z.)

Also solved by L. V. Mead, R. C. Yates, and the Proposer.

Function Having Constant Modulus on the Unit Circle

4467 [1951, 703]. *Proposed by D. J. Newman, Harvard University*

Prove that az^n is the only entire function whose modulus is constant for all $|z| = 1$.

Solution by Alex Rosenberg, University of Michigan. Let $f(z)$ be an entire function of constant modulus on $|z| = 1$, and denote the zeros of $f(z)$ within the unit circle by a_i, $i = 1, \cdots, k$. Now the function

$$g(z) = f(z) \prod_i \frac{(1 - \bar{a}_i z)}{(z - a_i)}$$

is an integral function of constant modulus on $|z| = 1$, and having no zeros in the unit circle. By applying the maximum modulus theorem for the unit disk to $g(z)$ and to $1/g(z)$ we see that $g(z) = \alpha$, a constant. Thus

$$f(z) = \alpha \prod_i \frac{(z - a_i)}{(1 - \bar{a}_i z)}.$$

But since $f(z)$ is an entire function, $a_i = 0$, so that $f(z) = \alpha z^n$.

It is clear that the same procedure will show that the most general meromorphic function of constant modulus on $|z| = 1$, is

$$\alpha \prod_i \frac{(z - a_i)}{(1 - \bar{a}_i z)} \prod_j \frac{(1 - \bar{b}_j z)}{(z - b_j)}.$$

Also solved by J. B. Kelly, Jacob Korevaar, M. S. Robertson, W. Seidel, Vidar Wolontis, and the Proposer.

Bounds for the Coefficient a_1 in Certain Series Expansions

4468 [1952, 45]. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be convergent for $|z|<1$, and further let $0<|f(z)|\leqq 1$ for $|z|<1$. Prove $|f'(0)|\leqq 2/e$.

Solution by M. S. Robertson, Rutgers University. Let

$$F(z)=\sum_{n=0}^{\infty}c_nz^n=-\log f(z),$$

$c_0=-\log a_0$, $c_1=-a_1/a_0$. Then $F(z)$ is regular and $RF(z)>0$ for $|z|<1$. Hence a well known inequality* gives $|c_1|\leqq 2Rc_0$. This gives $|a_1|\leqq 2|a_0|\log(1/|a_0|)\leqq 2/e$, since $|a_0|\leqq 1$. Further, $|a_1|=2/e$ when $f(z)=\eta\exp(\epsilon z-1)/(\epsilon z+1)$, for η, ϵ arbitrary complex numbers of absolute value one.

The following generalization can be proved: let $f(z)$ satisfy

1) $f(z)$ regular for $|z|<1$,
2) $|f(z)|\leqq 1$ for $|z|<1$,
3) $f(z)$ has precisely n zeros β_i, $i=1, 2, \cdots, n$, in $0<|z|<1$,
4) $f(0)\neq 0$.

Define μ and ν by

$$\mu=\sum_{i=1}^{n}\frac{1-|\beta_i|^2}{\beta_i},\qquad \nu=\prod_{i=1}^{n}\beta_i.$$

Then:

a) $|f'(0)|\leqq|\mu\nu|<1$ when $|\mu|\geqq 2$,
b) $|f'(0)|\leqq 2|\nu|e^{|\mu|/2-1}$ when $|\mu|\leqq 2$.

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and let $g(z)=\sum_{n=0}^{\infty}c_nz^n$ be defined by

$$f(z)=\prod_{i=1}^{n}\left(\frac{\beta_i-z}{1-\bar{\beta}_iz}\right)g(z).$$

Then $g(z)$ is regular for $|z|<1$, $|g(z)|\leqq 1$ for $|z|<1$, and when $a_0=f(0)\neq 0$ then $g(z)\neq 0$ for $|z|<1$. Thus $|c_1|\leqq 2|c_0|\log(1/|c_0|)$ as in the original problem. However,

$$a_1=\prod_{i=1}^{n}\beta_i\left[c_1-c_0\sum_{i=1}^{n}\frac{1-|\beta_i|^2}{\beta_i}\right],$$

$$|a_1|\leqq|\nu|\left[2|c_0|\log\frac{1}{|c_0|}+|\mu|\cdot|c_0|\right],\qquad |c_0|\leqq 1.$$

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* Carathéodory's inequality. See Nehari, *Conformal Mapping*, 1952, p. 170; also Bieberbach, *Funktionentheorie*, v. 2, p. 140.

For μ and ν fixed, if the right side is maximized over $0<|c_0|\leqq 1$, the stated results follow easily. Equality signs hold for

a) when $f(z)=\prod_{i=1}^{n}\left(\frac{\beta_i - z}{1-\bar{\beta}_i z}\right)$, $\quad 0<|\beta_i|<1,\ |\mu|\geqq 2;$

b) when $f(z)=\prod_{i=1}^{n}\left(\frac{\beta_i - z}{1-\bar{\beta}_i z}\right)\exp\left(\frac{|\mu|}{2}-1\right)\left(\frac{1+z}{1-z}\right)$,

$$0<|\beta_i|<1,\ |\mu|\leqq 2.$$

A slight modification takes care of the case when $f(0)=0$ instead of 4) above. Then $|f'(0)|\leqq|\nu|<1$, and equality holds when

$$f(z)=z\prod_{i=1}^{n}\left(\frac{\beta_i - z}{1-\bar{\beta}_i z}\right), \qquad 0<|\beta_i|<1.$$

The bounds obtained for $|f'(0)|$ are sharp and in each case less than unity. This is most easily seen from the inequality $|a_0|^2+|a_1|\leqq 1$ which holds for all $f(z)$ for which $|f(z)|\leqq 1$ in $|z|<1$. It would be an interesting problem to prove directly that the bound shown in b) is less than unity.

Also solved by J. B. Kelly, Jacob Korevaar, A. J. Lohwater, W. Seidel, Peter Ungar, Chih-yi Wang, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Theory of Matrices. By Sam Perlis. Cambridge, Mass., Addison-Wesley Press, Inc., 1952. xiv+237 pages. $5.50.

This is a finely written, excellent introduction to matrices. The style is natural and clear. Aimed at intelligent but immature students, a complete but elementary treatise on basic ideas was the author's purpose. In this he has succeeded. In subject material, we find deduction of canonical forms under equivalence, similarity (three different forms), congruence, Hermitian congruence, orthogonal and unitary similarity. Canonical forms for skew and skew-Hermitian matrices occur. These are not always to be found in such books. Matrix methods, exclusively, are employed. The elements are from a field (the only abstraction in the book) but may be taken as real or complex. There is a full treatment of polynomial matrices. The spectral theorem is treated well, including a clarifying preliminary step (9-5, p. 175) of identifying matrices susceptible

to spectral decomposition with those which are diagonable. Then, and earlier, characterizations of such matrices are given, culminating with admission of symmetric and Hermitian matrices, these being treated separately. The superdiagonal form is not given. Elementary divisor theory can be learned (pp. 124–158) with the vital fact (for teachers, and researchers seeking counterexamples) that a matrix exists with preassigned elementary divisors. There is also treatment of polynomials (unique factorization), vector space (as n-tuples), determinants (via elementary matrices). Linear transformations are postponed until the last chapter, and as a result similarity theory, spectral theorem *etc.*, are covered without the fundamental motivation. This, however, may be taken on faith, and the arguments have the advantage (for some) that they are purely arithmetic in nature. All treatments of transformations are based on coordinate systems. Inner products are touched lightly. The symbol (ξ, η) is not used; Schwarz's inequality does not appear; the adjoint of an operation is not defined, and for Hermitian matrices $(\xi, H\eta) = (H\xi, \eta)$ is not featured, but used once (p. 184) in the form $\bar{\xi}'H\eta = (\overline{H\xi})'\eta$.

There are appendices scattered with such topics as applications to differential equations and mechanics. Only the barest beginnings are made, however.

The treatment of each facet of a given topic is approached as the student is prepared for it so that there is scattering. For example, linear systems are discussed on pp. 45–7, 54, 83. Thus the value as a text exceeds the utility as a handbook. Also, in reading, I often had the feeling that a subject had been abandoned too soon, only to find it resumed later.

The method of exposition, according to the preface, consists of: " . . . coddling (of students) . . . supplemented by activities leading the student to stand on his own feet." We find instruction on pronunciation of $F[x]$ (p. 108) but not $F(x)$ (p. 122). We are exhorted to write I multiplying the constant term of a polynomial g, in forming $g(A)$, with two tellings (pp. 66–7, 136) and a warning in case of failure. We are warned of strange (p. 77) and unusual (p. 85) fields (those of characteristic 2). The author's solicitude leads him to commit the factorial howler (p. 85) of announcing that in these fields "$1+1=0$!" On the other hand, the language is concise; the statements of many theorems (*e.g.* 3-3, p. 42) will have to be restated with quantifiers before certain students will be able to read them.

Italics distinguish theorems and the first occurrence of words. Reference is aided by occurrence of chapter and section numbers on each page. There are a two-page summary and a bibliography at the end. Theorems and definitions are correctly stated and are self-contained. An important point!

Debits. It is not made clear that a polynomial is not a function with scalar values, *i.e.* that x is an "indeterminate" (p. 17). This is somewhat cleared up on p. 108, but is insufficiently emphasized, and, on p. 132, x is called a variable. The crux is that it is not made clear what equality of two polynomials means.

V_n is a set of n-tuples, V a sub-vector-space of V_n. Can V itself be considered as a V_k? Is a 5-sub-space of V_8 "like" V_5, or a 5-sub-space of V_7, or another

5-sub-space of V_6? These are unanswered and the idea "isomorphism" is missing.

The last four lines of p. 85, and p. 86 give the impression that quadrics with congruent matrices can be obtained from each other by a rigid motion.

On p. 125, we must select a "polynomial of minimal degree in the class of all elements of all matrices equivalent to A." This sounds hard. The author should mention that it is easy in practice.

There are no worked-out examples, the ends of proofs are not clearly marked; I dislike a numbering system that puts Lemma 4-2 after Theorem 4-5.

Index. Not all important mentions of a subject are indexed (*e.g.* diagonal 160, 171, 173, 174, 184), nor are items in problems (*e.g.* non-singular, 232). Omitted are proper matric polynomial, 132; square root, 203; trivial factor, 110. There is a lack of a symbol-index (*e.g.* $N(A)$, 53; N, 161; $O(M)$, 225).

Problems. These are good and cover the topics. They are often quite demanding. No answers are given. Number 3 on page 181 is so trivial ($A=0$) as to be confusing, and number 8 on page 174 is false, *e.g.*

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is not similar to a diagonal matrix.

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Calculus. By Tomlinson Fort. D. C. Heath and Co., 1951. xii+560 pages, $4.75.

Calculus (revised edition). By J. V. McKelvey. The Macmillan Co., 1951. vii+405 pages, $4.50.

Calculus. By J. F. Randolph. The Macmillan Co., 1952. x+483 pages.

Calculus. By A. H. Sprague. The Ronald Press Co., 1952. xi+576 pages. $6.50.

These four textbooks are designed for use in courses in calculus which presuppose a knowledge of plane analytic geometry. Fort and Sprague include a short but adequate discussion of polar coordinates. McKelvey's book also presupposes solid analytic geometry for some of the later chapters. The other three books give an adequate treatment for the purposes of calculus. All these texts give a rather complete treatment of more topics than can usually be covered in a course in differential and integral calculus. Fort and McKelvey include an extensive introduction to differential equations. No attempt will be made to compare these texts, but each has some essential characteristics that are worth pointing out to be interpreted as desired.

(1) Fort has given careful attention to mathematical details, proving most of the basic theorems carefully and clearly. Where proofs are omitted, he has tried to indicate what basic ideas are involved (*e.g.* that the proof of the existence of the limit of a bounded monotonic sequence depends on a systematic

study of the irrational numbers, a fact which is discussed further in the appendix). The first chapter is on infinite series, including the usual elementary tests for convergence and a brief treatment of power series. This is designed to develop the concept of limit and also to introduce power series for use in connection with exponential functions, trigonometric functions, and Taylor series. The power series definitions of sine and cosine are used along with the usual definition in deriving differentiation formulas for the trigonometric functions. He handles this and other seemingly advanced treatments in a way that should be convincing to elementary students and so that if not, the student can fall back on the usual treatment without being confused. The logarithm is introduced as an integral, its properties developed, and the exponential function defined in terms of the logarithm. Length of arc is defined as a definite integral before differentiation of arc length is discussed. This book attempts to handle calculus in a way unusual for American texts, but at the same time covers all the usual material with very clear, interesting, and concise explanations. For example, the concepts of slope, velocity and acceleration are introduced and carefully discussed as preparation for the definition of derivative. The attempt has not been so much to give a rigorous treatment as to make the ideas meaningful. It seems likely that he has succeeded. The appendix contains answers to most odd-numbered problems and tables of integrals, natural and Naperian logarithms, exponential and hyperbolic functions, and trigonometric functions.

(2) McKelvey's text is a revised edition of his calculus published in 1937. The general design of the book has not been changed, though an article on curvilinear motion and a chapter on limits and continuity have been added. The examples have been largely rewritten. They are particularly extensive and excellently chosen, with many examples to show the application of calculus in other fields. The concept of integration is not discussed until all the usual methods and applications of differential calculus have been developed, except for a short section in which arc length, plane areas, and volumes of revolution are computed as antiderivatives. Methods of evaluating indefinite integrals are treated extensively before the definite integral is introduced. Proofs of the fundamental limit and continuity theorems are omitted in the introductory chapter, the attempt being to concentrate on teaching understanding by use of examples. Careful definitions of limit of a sequence and continuity are given in the new chapter (Chapter 21), and a number of elementary theorems on limits are proven. Unfortunately, not so much care is given to the definition of limit of a function. Not only are such undefined concepts as "as x approaches c" used, but $\lim_{x\to c} f(x)$ is defined as $\lim_{n\to\infty} f(x_n)$ for any sequence $x_1, x_2, \cdots$ with $\lim_{n\to\infty} x_n = c$. The appendix contains a table of integrals and answers for all examples.

(3) Randolph has tried to write a text which "will provide some routine knowledge of the subject and its applications for all members of a class, but still challenge the good students." He does this largely in three ways: (1) Some advanced subjects such as line integrals and partial derivative systems are included. (2) Many explanations and proofs that might be omitted in a routine

course are italicised so omission does not seem confusing. Sometimes these are short remarks concerning rigor or lengthy discussions such as forms of the remainder in Taylor's theorem. (3) A number of subjects such as proofs of limit theorems and operations with power series are discussed carefully in the appendix with many interesting exercises given so that these topics may be taught along with the rest of the text. A careful discussion of tangents and velocity precedes the definition of derivative. The definite integral is introduced early. Many problems are given to emphasize the concept of definite integral as the limit of a sum. The definite integral is defined by using equal subdivisions and values at end-points, the usual definition being discussed in italics. This simplifies the definition, but complicates demonstrations of applications. An attempt is made to informally introduce foundations for several advanced topics. *E.g.* differential equations are introduced in connection with anti-derivatives, several simple differential equations are discussed along with methods of integration, and exact differential equations are introduced as "partial derivative systems." Great care is given to the function concept, the idea of ordered pairs being very carefully and clearly discussed. Some of the material in *Analytic Geometry and Calculus* by Randolph and Kac has been used, the books having great similarity in a few places. A table of integrals and "selected answers" are given.

(4) Sprague states that "the purpose of this book is to present the elements of the calculus as simply as possible without sacrificing rigor." In a number of cases (*e.g.* the fundamental limit theorems), proofs are given only after the theorems have been explained and used. An "intuitive" and the usual ϵ-δ definition of limit are given. The latter is used in proofs of limit and continuity theorems. Continuity is handled carefully throughout the text in a way that should be quite teachable. The definition of derivative is given and a number of elementary formulas developed before such usual aids to understanding as tangent lines and velocity are given. The anti-derivative is used briefly in connection with velocity and acceleration, but otherwise differential calculus is thoroughly studied before integration is introduced. Methods of evaluating indefinite integrals are thoroughly studied before the definite integral is defined as the limit of a sum. Some simple differential equations and applications are discussed in connection with indefinite integration. The definition of definite integral is very carefully and clearly given, with a discussion of uniform continuity and upper and lower integrals and a proof of the existence of the definite integral of a continuous function. The chapter on applications to solid analytic geometry includes an introduction to differential geometry. A short table of integrals, a collection of formulas, and answers to all problems are given.

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Symmetry. By Hermann Weyl. Princeton University Press, 1952. vii+168 pp. $3.75.

This little book, artistically printed and beautifully illustrated, consists of

four lectures given at Princeton as the author's "swan song" on the occasion of his retirement from the Institute for Advanced Study. It is so full of interesting material that, after beginning to read, one can hardly put it down. Although it is written for laymen, there are very few experts who will not learn something. For, besides mathematics and crystallography, it deals with philosophy, history, poetry, sculpture, painting, architecture, theology, physics, chemistry, physiology, embryology, and botany. The author has resisted the temptation to include an account of his own research on symmetry in space of many dimensions (see, *e.g.*, his contribution to the reviewer's *Regular Polytopes*, New York, 1949, pp. 204–207). Instead, he has collected a multitude of facts and theories, observations and quotations, drawings and photographs, blending them in his own inimitable fashion. The arrangement of text and illustrations is reminiscent of Steinhaus's *Mathematical Snapshots* (New York, 1950), but there is little overlapping of material.

The first lecture, on "Bilateral symmetry," contains an abundance of unusual information. We read, for example, that as many as one person in every five thousand is born with *situs inversus:* the consistent inversion of left and right throughout all the asymmetrical organs of the body. Turning from anatomy to biochemistry, the author gives a dramatic account of Pasteur's separation of laevo- and dextro-tartaric acid, of which only the latter occurs in organic nature. "Nature," he writes, "in giving us the wonderful gift of grapes so much enjoyed by Noah, produced only one of the forms, and it remained for Pasteur to produce the other!" But he soberly adds, "If there is a difference in principle between life and death, it does not lie in the chemistry of the material substratum."

The lecture on "Translatory, rotational, and related symmetries" includes a very readable introduction to the theory of vectors and to groups of transformations. On page 45, he makes the remark (too often overlooked in elementary instruction): "A vector is really the same thing as a translation, although one uses different phraseologies for vectors and translations." He ascribes to Leonardo da Vinci the enumeration of finite groups of congruent transformations in the plane: the cyclic groups C_1, C_2, $\cdots$ and the dihedral groups D_1, D_2, $\cdots$. These groups are illustrated by many examples from nature; *e.g.*, D_6 by snowflakes, D_5 by the geranium, and the rarer C_5 by *vinca herbacea*. At this point he might well have reproduced the 17-gonal section of a strand of equisetum from Hans Meierhofer, *Die Augen auf in unseres Herrgotts Garten!* (Zürich, 1947), page 48.

In connection with the occurrence of dihedral groups in animals, he refers to Sir D'Arcy W. Thompson's *Growth and Form*, "a masterpiece of English literature, which combines profound knowledge in geometry, physics, and biology with humanistic erudition and scientific insight of unusual originality."

One of the photographs reproduced from Thompson's book shows the giant sunflower, *Helianthus maximus*, whose florets form 34 right-handed and 55 left-handed spirals (p. 71). Since 34 and 55 are members of the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, · · · ,

this is an instance of the phenomenon called "phyllotaxis." A simpler instance is provided by any well-formed pineapple. The author cites an article of 1872 by P. G. Tait for the best attempt at an explanation.

He reproduces a page from Haeckel's *Challenger monograph*, which shows the skeletons of several Radiolarians, including an octahedron, an icosahedron and a dodecahedron "in astonishingly regular form." The same page had been used by Thompson (*Growth and Form*, Fig. 340), who made the following comment, in a letter to the reviewer, dated 9th March, 1947: "As to Haeckel, I wouldn't trust him round the corner, and I have the gravest doubt whether his pentagonal dodecahedron and various others ever existed outside his fertile fancy. I believe I may safely say that no type-specimens of these exist in the British Museum, or anywhere else. He was an artist, a pattern-designer, a skilled draughtsman. He had a minute professorial salary in a small University. The *Challenger* paid eight guineas apiece for as many plates as he chose to draw; and he kept on drawing them, and lived on the proceeds (so they used to say) till the end of his life. He represents a thoroughly bad period in Natural Science."

In the lecture on "Ornamental symmetry" we see that the plane tessellation of regular hexagons occurs not only on the floors of bathrooms but in the bees' honeycomb, in the parenchyma of maize, in the retinal pigment of our eyes, and on the surface of certain diatoms. There is an account of the enumeration (by Pólya and Niggli, 1924) of the seventeen two-dimensional crystallographic groups, which were unconsciously used by the Egyptians, Moors and Chinese in their ornaments. In this connection, some illustrations are reproduced from Owen Jones' *The Grammar of Ornament* (London, 1868). In the same lecture we find an introduction to the theory of positive definite quadratic forms and their representation by lattices. There is also a proof that the only possible periods for rotational symmetry of a lattice are 2, 3, 4, and 6. A still neater proof was given long ago by William Barlow, Philosophical Magazine (6 ser., 1, 1901, p. 17).

The final lecture, on "Crystals: the general mathematical idea of symmetry" begins with an outline of the enumeration of the 230 space-groups, which was carried out independently by Fedorov in Russia (1885), Schoenflies in Germany (1891) and Barlow in England (1894). Then the notion of a congruent transformation in Euclidean 3-space is extended to that of a Lorentz transformation in Minkowskian space-time. This "world" is an affine 4-space with a real cone of isotropic lines through each point. It is unfortunate that the author says (on p. 132): "The light cone . . . at a definite world-point O, 'here-now,' . . . divides the world into future and past." He seems to have overlooked the third region, exterior to the cone, which one could reasonably call "the present."

From relativity theory he deftly turns to quantum theory, Galois theory, and cyclotomy, showing the importance of the group of automorphisms in the investigation of any "structure-endowed entity."

The author's command of language is truly amazing. Who else would say (p. 127), "Temperature is the environmental factor *kat'exochen*"? Only twice is he at a loss for the English equivalent of a German word: on page 77 he writes "Umklappung" for *half-turn*, and on page 96 "modul" for *modulus*.

The four lectures are supplemented by two mathematical appendices: "Determination of all finite groups of proper rotations in 3-space" and "Inclusion of improper rotations." The latter is especially neat. Finally, there is a list of acknowledgments for the 72 figures, and a good index (except that the entry "Thompson" confuses Sir D'Arcy with Lord Kelvin, whose space-filling of truncated octahedra is displayed on page 92).

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NEW BOOKS RECEIVED

Basic Skills in Mathematics. By H. V. Price and L. A. Knowler. Boston, Ginn and Company, 1952. 8+249 pages. $3.25.

Light (The Student's Physics, Vol. I.). By R. W. Ditchburn. Glasgow, Scotland, Blackie and Son, Ltd., 1952. 22+680 pages.

Science and Method. By Henri Poincaré. New York, Dover Publications, Inc., 1952. 288 pages. Paperbound, $1.25; Clothbound, $2.50.

Science and Hypothesis. By Henri Poincaré. New York, Dover Publications, Inc., 1952. 244 pages. Paperbound $1.25.

Tables of the Bessel Functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, 0-x-1, Washington, D. C., National Bureau of Standards. October, 1952. 60 pages, 40 cents.

Vlakke Meetkunde voor Voortgezette Studie. By P. Wijdenes. Groningen, Djakarta, P. Noordhoff, 1952. F 13.00, geb f14.50. 303 pages.

Statistical Theory in Research. By R. L. Anderson and T. A. Bancroft. New York, McGraw-Hill, 1952. xvi+399 pages, $7.00.

Methods of Statistical Analysis, 2nd Edition. By C. H. Goulden. New York, John Wiley and Sons, Inc., 1952. vi+467 pages. $7.50.

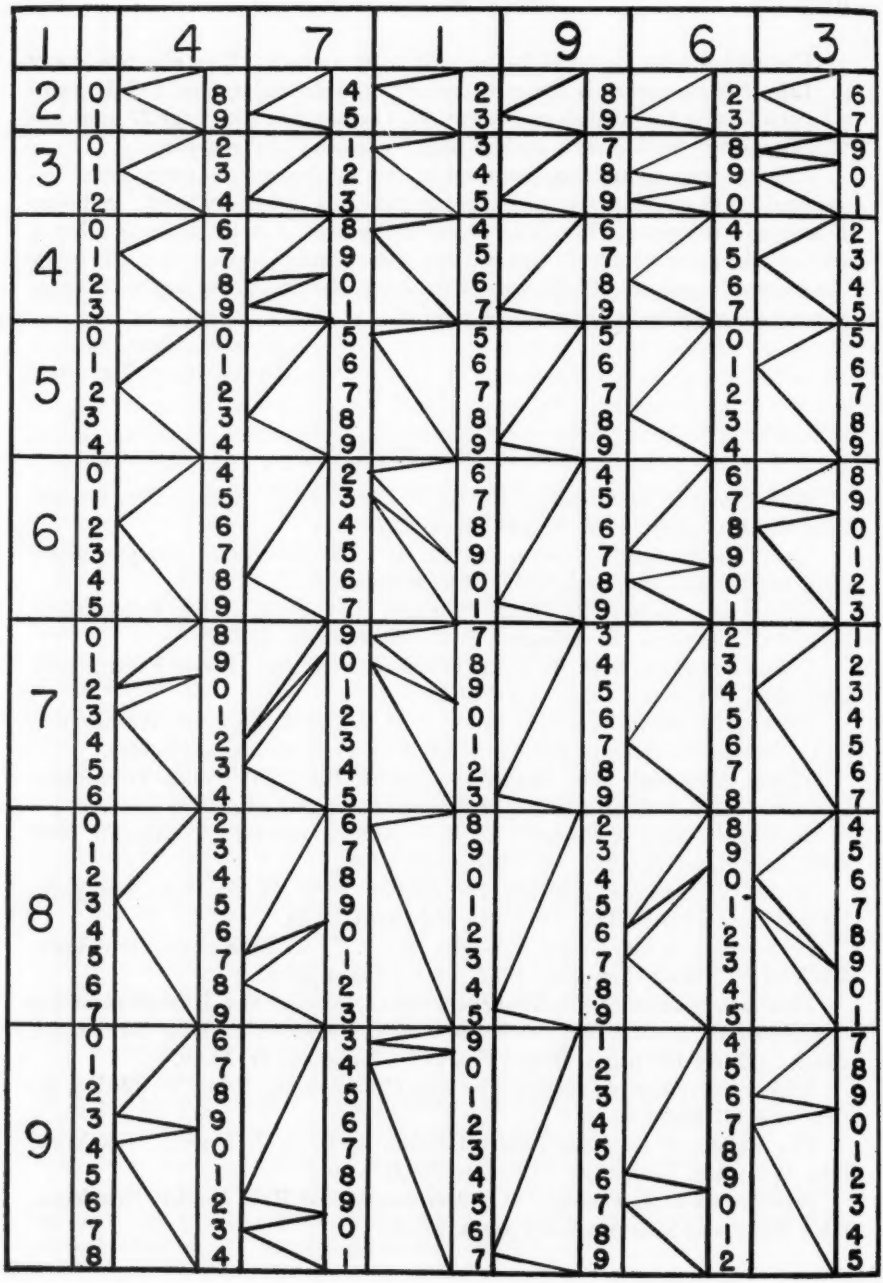
Introduction to the Theory of Games. By J. C. C. McKinsey. New York, McGraw-Hill Book Company, 1952. x+371 pages. $6.50.

Theorie der Geometrischen Konstruktionen (*Lehrbucher und Monographien aus dem Gebiete der Exakten Wissenschaften*). By L. Bieberbach. Verlag Birkhauser, Basel, 1952. 6+162 pages. Bound Frs. 18.70, Unbound Fr. 15.60.

Society of Actuaries' Textbook on *Life Contingencies*. By C. W. Jordan, Jr. 1952. xi+331 pages. $8.00.

The Methods of Statistics. Fourth Edition. By L. H. C. Tippett. New York, John Wiley and Sons, 1952. 395 pages. $6.00.

Introduction to the Foundations of Mathematics. By R. L. Wilder. New York, John Wiley and Sons. xiv+305 pages. $5.75.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CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

GENAILLE'S RODS

Always a favorite topic for club discussions is that of "Napier's Rods," a device for reducing multiplication to a series of additions. Napier describes these well-known rods, which became popularly known as Napier's "bones," in his *Rabdologia* published shortly after his death in 1617. The literature contains several accessible articles describing these bones.* Not so well-known are the more ingenious rods invented by Genaille in 1885. Readers familiar with Napier's bones should have no difficulty in understanding the principle underlying Genaille's rods upon studying the figure on the opposite page. Here the rods are placed for the multiplicand 471,963; the partial products are read from right to left, the points of the triangles guiding the eye to the proper figures.

* For example, see *The Pentagon*, Spring 1949, pp. 98–100.

MATHEMATICS AND PHILATELY

An interesting club project would be the collecting of postage stamps which have reference in some way to mathematics or mathematicians. For example, a topical collection could be formed of stamps portraying geometrical figures such as triangles, pentagons, octagons, *etc.* Stamps which carry portraits of mathematicians are of particular interest. A comprehensive description of such stamps was published by C. B. Boyer in 1949.* Since that time, several additional mathematicians have been so honored, and a revised list might be of interest.

The following check-list contains 82 items honoring 34 mathematicians. Catalog numbers and descriptions given are those in *Scott's Standard Postage Stamp Catalogue*, 1952.

Abel, Niels Henrik (1802–1829)
Norway, 1929:
145. 10 ö green
146. 15 ö red
147. 20 ö chocolate
148. 30 ö ultramarine

Ampere, Andre Marie (1775–1836)
France, 1936:
306. 75 c brown
France, 1949:
626. 15 fr sepia

Armero, Julio Garavito (1865–1920)
Colombia, 1949:
573. 4 c green on white

Avicenna (979–1037)
Lebanon, 1948:
223. 30 pi orange brown and buff
224. 40 pi Prussian green and buff
Germany, 1952:
10N106. 35 pf blue

Bolyai, Farkas (1775–1856)
Hungary, 1932:
479. 70 f cerise

* Carl B. Boyer, "Mathematicians and Philately," *Scripta Mathematica*, vol. 15, pp. 105–114, June, 1949.

Boscovich, Roger Joseph (1711–1787)
Croatia, 1943:
59. 3.50 k copper red
60. 12.50 k dark violet brown
Brahe, Tycho (1546–1601)
Denmark, 1946:
300. 20 ö dark red
Chaplygin, Sergei A. (1869–1944)
Russia, 1944:
945. 30 k gray
946. 1 r light brown
Chebychev, Pafnutiy Lvovich (1821–1894)
Russia, 1946:
1050. 30 k brown
1051. 60 k gray-brown
Condamine, Charles-Marie de la (1701–1774)
Ecuador, 1936:
347. 2 c blue
348. 5 c olive-green
349. 10 c orange
350. 20 c purple
351. 50 c salmon
Copernicus, Nicholas (1473–1543)
Poland, 1923:
192. 1000 m indigo
193. 5000 m rose
Poland, 1942–43:
NB23. 1 z+1 z dull myrtle green
NB27. 1 z+1 z rose lake
Descartes, René (1596–1650)
France, 1937:
330. 90 c copper red
331. 90 c copper red
Dürer, Albrecht (1471–1528)
Germany, 1925:
362. 80 pf chocolate
Eötvös, Lorand (1848–1919)
Hungary, 1932:
471. 6 f yellow green
Hungary, 1948:
840. 60 f deep red
Euler, Leonard (1707–1783)
Germany, 1950:
10N58. 1 pf gray
Galilei, Galileo (1564–1642)
Italy, 1933:
D16. 35 c rose red
Italy, 1942:
419. 10 c dark orange and lake
420. 25 c gray green and green
421. 50 c brown violet and violet
422. 1.25 l Prussian blue and ultramarine
Italy, 1945:
D18. 1.40 l blue
Gerbert, i.e., *Pope Sylvester II* (c. 950–1003)
Hungary, 1938:
511. 1 f deep violet
516. 10 f red orange
Gazeta, Matematica
Romania, 1945:
596. 2 l sepia
597. 80 l blue-black
Hamilton, Sir William Rowan (1805–1865)
Irish Free State, 1943:
126. ½ d green
127. 2½ d brown
Huygens, Christiaan (1629–1695)
Netherlands, 1928:
B36. 12½ c+3½ c ultramarine
Jacobsen, Jacob Christian (1811–1887)
Denmark, 1947:
304. 20 ö dark red
Leibniz, Gottfried Wilhelm (1646–1716)
Germany, 1926:
360. 40 pf deep violet
Lorentz, Hendrik Antoon (1853–1928)
Netherlands, 1928:
B35. 7½c+3½c vermillion
Mercator, Gerard (1512–1594)
Belgium, 1942:
B324. 1.75 fr+50 c dull blue
Ortelius, Abraham (1527–1598)
Belgium, 1942:
B325. 3.25 fr+3.25 fr lilac rose
Pascal, Blaise (1623–1662)
France, 1944:
B181. 1.20 fr+2.80 fr black
Poincaré, Henri (1854–1912)
France, 1952:
18 fr+5 fr black
Roemer, Ole (1644–1710)
Denmark, 1944:
293. 20 ö henna brown
Stevin, Simon (1548–1620)
Belgium, 1942:
B321. 50 c+10 c fawn
Swedenborg, Emanuel (1688–1772)
Sweden, 1938:
264. 10 ö violet
266. 10 ö violet
267. 100 ö green
Vasilyevich, Mihail (1801–1861)
Russia, 1951:
1604. 40 k black brown on pink

Vinci, Leonardo da (1452–1519)
Italy, 1932:
C29. 1 l violet
C30. 3 l brown red
C31. 5 l deep green
C33. 10 l+2.50 l black brown
C34. 100 l bright blue and greenish black
Italy, 1935:
347. 50 c purple
348. 1.25 l dark blue
Italy, 1937:
404. 50 c light violet
C103. 2 l royal blue
C105. 5 l deep green
Liechenstein, 1948:
C24. 10 rp dark green
France, 1952:
682. 30 fr deep ultramarine
Germany, 1952:
10N104. 20 pf green
Hungary, 1952:
C109. 1.60 fo dark Prussian green
Italy, 1952:
601. 25 l deep orange
Poland, 1952:
B73. 30 g+15 g ultramarine
Romania, 1952:
878. 55 b purple
Witt, Johan de (1625–1672)
Netherland, 1947:
B177. 7½ c+2½ c dark purple brown
Zhukovsky, Nikolai (1847–1921)
Russia, 1941:
838. 15 k deep blue
839. 30 k carmine rose
840. 50 k brown violet
Russia, 1947:
1098. 30 k sepia
1099. 60 k blue violet

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

EUROPEAN STUDY TRIP FOR MATHEMATICS AND SCIENCE TEACHERS

The School of Education of Boston University is sponsoring a study trip for approximately ten weeks during the summer of 1953. This trip is designed to acquaint mathematics and science teachers with the background and resources of Western Europe which will be applicable to their teaching. England, Scotland, Norway, Sweden, Denmark, the Netherlands, Germany, Switzerland, and France will be visited. Six semester hours of college credit will be credited through Boston University.

The group will be accompanied by Professor H. W. Syer of Boston University who will act as director of the tour and guide to the cities. Plans will be made in advance for the group to be met by mathematics and science teachers from the country being visited, officials from the ministries of education, and experts of the museums and laboratories so that authentic information will be at hand. A specialized guidebook entitled "Resources in Europe for Mathe-

matics and Science Teachers" has been prepared.

For further information write to Professor H. W. Syer, School of Education, 332 Bay State Road, Boston, Massachusetts.

PERSONAL ITEMS

Professor W. H. Fagerstrom of City College of the City of New York represented the Association at the Seventeenth Educational Conference which was held in New York City on October 30–31, 1952.

Professor Einar Hille of Yale University was a representative of the Association at the first Assembly of the International Commission for Mathematical Instruction which was held at Geneva, Switzerland, on October 20–21, 1952.

The honorary degree of Doctor of Science was conferred on Dean Emeritus L. P. Eisenhart by Princeton University on June 17, 1952.

Dr. Kurt Gödel of the Institute for Advanced Study was awarded the honorary degree of Doctor of Science by Harvard University on June 19, 1952.

Dr. A. E. Livingston of the University of Oregon has received a National Science Foundation Postdoctoral Fellowship and is at the Institute for Advanced Study.

Associate Professor V. O. McBrien who has received a Ford Foundation Fellowship is on leave of absence from the College of the Holy Cross and is at Harvard University.

Mr. E. C. Molina of Newark College of Engineering was awarded the Cressan Medal by the Franklin Institute, Philadelphia, Pennsylvania, on October 15, 1952, in recognition of his contributions to telephony.

Professor R. C. Stephens of Knox College has been awarded a Ford Foundation Faculty Fellowship and is spending the year at Princeton University.

Alabama Polytechnic Institute announces: Mr. J. A. Pond has been appointed to an assistant professorship; Miss Frances A. Norton has been appointed to an instructorship.

At Boston College: Dr. Lorenzo Calabi, previously at the University of Rome, has been appointed to an assistant professorship; Reverend J. J. McCarthy, formerly professor of Physics and Mathematics at Weston College, has been appointed to an associate professorship; Mr. P. T. Banks has been promoted to an assistant professorship.

Bradley University reports the following: Mr. J. R. Brown, previously a graduate student at Kansas University, and Mr. H. E. Sandstrom, who has been a graduate student at Northwestern University, have been appointed to assistant professorships; Mr. J. H. Hafferkamp, formerly a graduate student at Bradley University, has been appointed to an instructorship.

Brown University makes the following announcements: Dr. W. G. Lister, who was AEC Postdoctoral Fellow at Yale University during 1951–52, has been appointed to an assistant professorship; Professor R. E. Gilman is on leave of absence until June 30, 1953 and is with the Weapons Systems Evaluation Group, Washington, D. C.; Professor Maurice Heins, who is on sabbatical leave during

1952–53, has a Fulbright award for research and is spending the year in Paris, France; Professor Heins was a delegate of the American Mathematical Society to the International Congress of Mathematicians at Salzburg, Austria, in September, 1952.

At Carnegie Institute of Technology: Associate Professor E. A. Whitman has retired with the title of Associate Professor Emeritus; Mr. C. E. Lemke, Mr. F. B. Smith, and Mr. S. C. Ying have been appointed to instructorships.

Catholic University of America announces: Dr. W. W. Boone has been promoted to an assistant professorship; Dr. R. J. Silverman of the University of Illinois has been appointed to an instructorship.

Johns Hopkins University makes the following announcements: Assistant Professor G. D. Mostow of Syracuse University and Dr. N. C. Ankeny, previously a member of the Institute for Advanced Study, have been appointed to assistant professorships; Dr. T. J. Rivlin, formerly a graduate student at Harvard University, has been appointed to an instructorship; Instructor A. B. J. Novikoff is now with the Institute for Cooperative Research of the University.

Knox College reports the following: Assistant Professor A. O. Lindstrum is Acting Chairman of the Department of Mathematics during the absence of Professor R. C. Stephens; Dr. I. J. Christopher of the University of Oregon has been appointed to an instructorship.

Los Angeles City College announces that the third annual William B. Orange Mathematics Prize Competition for students in the high schools of the Los Angeles School District will be held in May, 1953; one hundred fifty-five students participated in the preceding contest and prizes were awarded to ten of the thirty-two participating teams and to twenty-four individuals.

Montana State College reports: Assistant Professor Bernard Ostle of Iowa State College has been appointed to an associate professorship; Dr. Frederick Young, previously with the United States Naval Ordnance Test Station, China Lake, California, has been appointed to an assistant professorship; Mr. Glen Ingram, formerly a student at the College, has been appointed to an instructorship; Mr. J. E. Whitesitt has been awarded a predoctoral National Science Research Fellowship and is spending the year at the University of Illinois.

North Carolina State College announces the promotions of Miss Anna M. Harris and Mr. C. F. Lewis to assistant professorships.

Reed College announces the following: Dr. Tong Hing of Columbia University has been appointed to an assistant professorship; Mr. Joseph Roberts of the University of Minnesota has been appointed to an instructorship.

At Southern Illinois University Dr. Annette Sinclair of the University of Tennessee has been appointed to an assistant professorship. Professor H. S. M. Coxeter of the University of Toronto was one of the four guest lecturers at the Leonardo da Vinci festival which was held at the University during November 12–25, 1952; he discussed the application of mathematics in the work of da Vinci.

University of Georgia announces: Professor Tomlinson Fort has retired as

Head of the Department of Mathematics but retains the position of Regents' Professor of Mathematics; Professor Fort is on sabbatical leave during 1952–53 and is lecturing at the Royal Naval College, University of Aberdeen, Cambridge University and other British universities; Associate Professor A. C. Cohen, Jr. has been promoted to a professorship; Dr. S. E. Dyer has been appointed to an assistant professorship. Professor W. T. Martin of Massachusetts Institute of Technology gave a series of lectures on Analytic Functions of Several Complex Variables at the Colloquium of the University on November 19–21, 1952.

University of Mississippi reports the following: Mr. J. C. McCall, previously a graduate student at the University, has been appointed to an instructorship; Assistant Professor R. D. Sheffield has been granted a leave of absence for an additional year to continue graduate study.

At the University of Nebraska: Assistant Professor O. C. Collins has been promoted to the position of Associate Professor of Astronomy; Assistant Professor W. G. Leavitt has been promoted to an associate professorship; Dr. T. A. Newton, previously a teaching assistant at the University of Georgia, and Mr. A. J. Tingley, formerly a teaching assistant at the University of Minnesota, have been appointed to instructorships.

Dr. M. A. A. Al-Bassam has been promoted to a professorship at Huston-Tillotson College.

Dr. P. H. Anderson has accepted a position as a price economist with the Office of Price Stabilization, Washington, D. C.

Mr. D. L. Arenson who has been with the Armour Research Foundation has a position as Senior Research Engineer with Cook Research Laboratory, Skokie, Illinois.

Mr. B. J. Ball of the University of Texas has been appointed to an acting assistant professorship at the University of Virginia.

Associate Professor R. H. Bardell of the University of Wisconsin in Milwaukee has been promoted to the position of Professor and Chairman of the Department of Mathematics.

Mr. William Beck, formerly a graduate assistant at the University of Kentucky, has been appointed to a teaching assistantship at the University of Southern California.

Professor H. A. Bernhard of the Evans Signal Corps Laboratory has accepted a position as Senior Engineer with the Curtiss-Wright Corporation.

Mr. R. E. Blasch, previously a student at Hofstra College, has been appointed to an assistantship at the University of Illinois.

Assistant Professor J. H. Blau of Pennsylvania State College has been appointed to an associate professorship at Antioch College.

Mr. Daniel Block of Yeshiva University has been promoted to an assistant professorship.

Professor N. R. Bryan of the University of Maine has retired with the title of Professor Emeritus.

Reverend W. F. Burns, formerly of the College of the Holy Cross, is now at Fairfield University.

Assistant Professor L. J. Burton of Bryn Mawr College has been appointed to an assistant professorship at Lake Forest College.

Dr. R. K. Butz of the University of Georgia has been appointed to an assistant professorship at Colorado Agricultural and Mechanical College.

Miss F. Marion Clarke, formerly at the University of Nebraska, is now at Aberdeen Proving Ground, Maryland.

Mr. D. L. Daly has been appointed to an instructorship at Missouri School of Mines and Metallurgy.

Mr. D. J. Davis, formerly a mathematician in the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland, has accepted a position as a physicist with the Rocket Development Group, Redstone Arsenal, Huntsville, Alabama.

Mr. R. C. Davis of the University of Akron has been promoted to an assistant professorship.

Professor Patrick Du Val of the University of Georgia is now at the University of Bristol, England.

Graduate Assistant E. R. Epperson of Miami University has a position as a graduate fellow at Massachusetts Institute of Technology.

Associate Professor Ky Fan of the University of Notre Dame has been promoted to a professorship.

Mr. G. C. Francis, previously at Carleton College, has accepted a position as a mathematician with Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mrs. Ruth M. Frisch who has been a graduate student at the Polytechnic Institute of Brooklyn has been appointed to a graduate assistantship at Syracuse University.

Mr. H. N. Garber, previously a student at the University of Pennsylvania, has been awarded a National Science Foundation Fellowship and is a graduate student at Massachusetts Institute of Technology.

Assistant Professor W. H. Gottschalk of the University of Pennsylvania has been promoted to an associate professorship.

Dr. Emil Grosswald of the University of Saskatchewan has been appointed to the rank of associate at the University of Pennsylvania.

Dr. Carl Hammer, formerly chairman of the Division of Technical Education, Walter Hervey Junior College, New York City, has accepted a position as Senior Research Engineer at Franklin Institute, Philadelphia, Pennsylvania.

Mr. J. S. Hokanson, previously of Ripon College, has a position with Oscar Mayer and Company, Madison, Wisconsin.

Assistant Professor C. E. Jones of Agricultural and Technical College of North Carolina has been appointed to an associate professorship in the Engineering School of Tennessee Agricultural and Industrial State College.

Dr. S. T. Kao, previously a graduate student at Catholic University of

America, has been appointed to an assistant professorship at the College of St. Joseph on the Rio Grande, Albuquerque, New Mexico.

Associate Professor Leo Katz who is on leave of absence from Michigan State College during 1952–53 is at the Statistical Laboratory, University of California, Berkeley.

Assistant Professor Dora E. Kearney of Iowa State Teachers College, Cedar Falls, has been appointed to an assistant professorship at Westminster College, Salt Lake City, Utah.

Dr. H. W. Kuhn, previously lecturer at Princeton University, has been appointed to an assistant professorship at Bryn Mawr College.

Dr. J. R. Larkin of the University of Kansas has accepted a position as a mathematician with the Bureau of Ordnance, Navy Department, Washington, D. C.

Dr. A. E. Livingston, formerly a graduate assistant at the University of Oregon, is now a member of the Institute for Advanced Study.

Assistant Professor G. R. MacLane of Rice Institute has been promoted to an associate professorship.

Associate Professor Leonard McFadden of Virginia Polytechnic Institute has been promoted to a professorship.

Dr. Paul Meier has accepted a position as a research associate at the School of Hygiene and Public Health, Johns Hopkins University.

Mr. E. J. Musch who has been an instructor at the University of Louisville is now a mathematician at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Professor E. N. Oberg, who was on leave of absence during the academic year 1951–52 to serve as an applied mathematician with the North American Aviation, Incorporated, Los Angeles, California, has resumed his duties in the Department of Mathematics and Astronomy at the State University of Iowa.

Mr. T. J. Pignani, previously a graduate student at the University of North Carolina, has been appointed to an instructorship at Loyola University, New Orleans, Louisiana.

Miss V. Elise Qualls, formerly an instructor at Tennessee Polytechnic Institute, has a position as a physicist at the Ballistic Laboratory, E. I. du Pont de Nemours and Company, Wilmington, Delaware.

Mr. A. A. Ritchie who has been teaching at Ponca City Senior High School, Oklahoma, has been appointed to an assistant professorship at Millsaps College.

Mr. P. T. Rotter of the Mutual Benefit Life Insurance Company has been promoted to Associate Mathematician.

Assistant Professor Arthur Saastad of DePaul University has a position as an industrial engineer with United States Steel Company, Chicago, Illinois.

Assistant Professor D. R. Scholz of Southwestern Louisiana Institute has been appointed to an assistant professorship at Louisiana State University.

Dr. R. W. Shephard of the Rand Corporation has accepted a position with the Sandia Corporation, Albuquerque, New Mexico.

Professor L. W. Sheridan is on leave of absence from the College of St. Thomas for the year 1952–53 and is spending this year at the Aeronautical Division of the Minneapolis Honeywell Regulator Company as a research engineer.

Dr. C. B. Tompkins of George Washington University has accepted a position as a mathematician at the National Bureau of Standards, Los Angeles, California.

Mr. J. P. van Alstyne of Hamilton College has been promoted to an assistant professorship.

Mr. J. E. Vollmer of the University of Vermont has been appointed to an instructorship at the University of Detroit.

Mr. E. A. Voorhees, Jr., has accepted a position as a staff member at Los Alamos Scientific Laboratory.

Mr. G. P. Weig of St. Ambrose College has been appointed to a graduate assistantship at Iowa State College.

Mr. R. S. Wollan, previously a graduate student at the University of Georgia, has been appointed to an instructorship at Ely Junior College, Minnesota.

Dr. R. K. Zeigler has accepted a position as a statistician with Los Alamos Scientific Laboratory.

Professor Emeritus G. N. Bauer of the University of New Hampshire died on October 13, 1952. He was a charter member of the Association.

Mr. F. F. Dorsey of South Orange, New Jersey, died in June, 1952.

Professor M. A. Nordgaard of Upsala College died on October 18, 1952.

Mr. G. K. Rosenblum who was a student at the University of Pennsylvania died on October 2, 1952.

Mr. W. L. Sauter who was a student at the University of Louisville died on June 7, 1952.

Associate Professor E. D. Wells of the University of Pittsburgh died on August 13, 1952. He had been a member of the Association for twenty-six years.

ACKNOWLEDGEMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: Arnold, B. H.; Ball, R. W.; Ballantine, J. P.; Bateman, P. T.; Beaumont, R. A.; Byrne, W. E.; Carver, W. B.; Chapman, D. G.; Chowla, S.; Coe, C. J.; Court, N. A.; Coxeter, H. M. S.; Dekker, D. B.; Ford, L. R.; Forsythe, G. E.; Frame, J. S.; Frink, O.; Gale, David; Gleason, A. M.; Herskovits, M. J.; Hewitt, E.; Householder, A. S.; Jones, P. S.; Kaplansky, I.; Kelley, J. L.; Kuhn, H. W.; Lax, P. B.; Leipnik, R. B.; McCoy, N. H.; McEwen, D. M.; McFarlan, L. H.; MacLane, S.; May, K. O.; Melahn, W.; Ore, O.; Pall, G.; Randolph, J. F.; Rauch, L. E.; Redheffer, R. M.; Robinson, R.;

Rosenbloom, P. C.; Rosser, J. B.; Seebeck, C. L.; Sheffer, I. M.; Springer, C. E.; Struik, D. J.; Vaughan, H. E.; Wade, T. L.; Wallace, A. D.; Winger, R. M.; Yates, R. C.; Zuckerman, H. S.

Mathematical Notes: Agnew, R. P.; Beckenbach, E. F.; Berkowitz, J.; Blank, A.; Blumenthal, L. M.; Brauer, A. T.; Brenner, J. L.; Boas, R. P.; Busemann, H.; Carlitz, L.; Carver, W. B.; Craig, H. V.; Erdelyi, A.; Feller, W.; Fine, N. J.; Frame, J. S.; Friedman, B.; Grove, V. G.; Gustin, W. S.; Hall, M.; Hanson, E. H.; Harrington, W. J.; Herzog, F.; Hollcroft, T. R.; Isaacson, E.; Kaplansky, I.; Lax, P.; Lee, H. L.; Lehmer, D. H.; Levine, J.; MacDuffee, C. C.; Mann, H. B.; Mann, W. R.; Murnaghan, F. D.; Nirenberg, L.; Parker, W. V.; Pettis, B. J.; Price, G. B.; Rainville, E. D.; Reid, W. T.; Rinehart, R. F.; Robinson, R.; Salzer, H. E.; Shapiro, H.; Sinclair, A.; Snyder, W. S.; Struik, D. J.; Tornheim, L.; Truesdell, C. A.; Wilansky, A.; Wilson, R. L.; Wright, E. M.; Zygmund, A.

Classroom Notes: Apostol, T. M.; Bryan, J. G.; Carver, W. B.; Coddington, E. A.; Cohen, I. S.; Corliss, J. J.; Fort, M. K.; Franklin, Philip; Harrington, W. J.; Hart, J. J.; Hildebrand, F. B.; Huff, G. B.; McCoy, N. H.; Ogilvy, C. S.; Parker, W. V.; Richmond, D. E.; Rudin, Walter; Singer, I. M.; Spear, Joseph; Struik, D. J.

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, Canada, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS, Washburn Municipal University of Topeka, April 11, 1953.

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Teachers College, Columbia University, March 28, 1953.

MICHIGAN, Wayne University, Detroit, April 18, 1953.

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, April 24, 1953.

NEBRASKA

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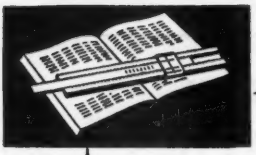

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